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J. Phys. A: Math. Theor. **40** (2007) 4691–4716 doi:10.1088/1751-8113/40/18/002

A supersymmetry approach to almost diagonal random matrices

Oleg Yevtushenko[1] **and Alexander Ossipov**[2]

[1] The Abdus Salam ICTP, Strada Costiera 11, I-34014 Trieste, Italy
[2] Instituut-Lorentz, Universiteit Leiden, PO Box 9506, 2300 RA Leiden, The Netherlands

E-mail: bom@ictp.it and ossipov@lorentz.leidenuniv.nl

Received 18 January 2007, in final form 19 March 2007
Published 17 April 2007
Online at stacks.iop.org/JPhysA/40/4691

Abstract

We develop a supersymmetric field-theoretical description of the Gaussian ensemble of the almost diagonal Hermitian random matrices. The matrices have independent random entries $H_{i\geqslant j}$ with parametrically small off-diagonal elements $H_{ij}/H_{ii} \sim \mathcal{B} \ll 1$. We derive a regular virial expansion of correlation functions in the number of 'interacting' supermatrices associated with different sites in the real space and demonstrate that the perturbation theory constructed in this way is controlled by a small parameter $\mathcal{B}$. The general form of the integral expression for the mth virial coefficient governed by the 'interaction' of m supermatrices is presented and calculated explicitly in the cases of 2- and 3-matrix 'interaction'. The suggested technique allows us to calculate both the spectral correlations and the correlations of the eigenfunctions taken at different energies and in different space points.

PACS numbers: 02.10.Yn, 71.23.An, 71.30.+h, 71.23.−k

1. Introduction

1.1. Conventional and unconventional random matrix theories

The random matrix theory (RMT) is a very useful mathematical formalism which allows us to describe universal properties of complex quantum systems. Let us consider an ensemble of $N \times N$ Hermitian matrices, whose elements are independent Gaussian-distributed random variables with a zero mean value and a position dependent variance:

$$\langle H_{ij} \rangle = 0; \qquad \left\langle H_{ii}^2 \right\rangle = \frac{1}{\beta}, \qquad \langle |H_{i\neq j}|^2 \rangle = \frac{1}{2}\mathcal{B}^2 \mathcal{F}(|i-j|). \tag{1}$$

Here $\langle \cdots \rangle$ denotes averaging over different realizations of RMs; the parameter β corresponds to the Wigner–Dyson symmetry classes: $\beta = 1$ for the Gaussian orthogonal ensemble (real

matrices) and $\beta = 2$ for the Gaussian unitary ensemble (complex matrices). The function $\mathcal{F}$ and the parameter $\mathcal{B}$ determine various universality classes sharing the same global symmetries.

A special case of the constant variance of the off-diagonal elements $\mathcal{B}^2\mathcal{F}(|i-j|) = 1$ corresponds to the archetypal Wigner–Dyson RMT [1]. It has a great number of applications starting from nuclear physics to quantum chaos to mesoscopic physics [2, 3]. Ergodic wavefunctions and a level repulsion are essential features of the Wigner–Dyson RMT.

Recently, *unconventional* RMTs characterized by decreasing function $\mathcal{F}$ have attracted a substantial interest (see, for instance, detailed introductions in [4–6]). This interest is stimulated, in particular, by a possibility of exploring the properties of localized and critical disordered systems. For example, if the off-diagonal matrix elements are essentially non-zero only inside a band centred at the main diagonal and decay exponentially fast to zero outside the band, all eigenfunctions are exponentially localized. This *banded* RMT describes the physics of a quasi-one-dimensional disordered wire [7]. If $\mathcal{F}$ decays only as a power-law outside the band [8], $\mathcal{F} \sim 1/|i-j|^{2\alpha}$, the eigenfunctions are power-law localized for $\alpha > 1$. The Wigner–Dyson universality class is approached for $\alpha \to 0$. Thus, the *power-law-banded* RMT (PLBRM) can interpolate between the Wigner–Dyson statistics and the Poisson statistics of localized system. The special case $\alpha = 1$ corresponds to a critical behaviour similar to that found at the point of the Anderson metal–insulator transition [8–10]. The function $\mathcal{F}_{\rm crit}$ for the one-parametric family of the critical PLBRM can be defined as follows: $\mathcal{F}_{\rm crit}(|i-j|) = 1/(\mathcal{B}^2+(i-j)^2)$. The eigenfunctions of this critical model remain multifractal at any $\mathcal{B}$ ranging from the weak multifractality at the large bandwidth $\mathcal{B} \gg 1$ to the strong multifractality for the *almost diagonal* RMT $\mathcal{B} \ll 1$.

1.2. From large to small bandwidth RMT: σ-model versus virial expansion

The considerable progress in the banded and the power-law-banded RMT has become possible due to the mapping [7, 8] onto the nonlinear supersymmetric σ-model [11], which is a powerful filed-theoretical description of various averaged correlation functions. However, such a mapping is only justified in the large bandwidth limit. This limitation comes from the saddle-point approximation which is a crucial step in the derivation of the σ-model. Physically, it corresponds to the diffusive approximation which implies in particular that the smooth envelope of a typical eigenfunction changes slowly on the scale of the mean free path. This approximation fails in the opposite limit of the small bandwidth, including the case of the *almost diagonal RMTs* where the bandwidth shrinks to zero and the off-diagonal matrix elements are assumed to be parametrically smaller than the diagonal ones, $\mathcal{B} \ll 1$.

Let us represent the almost diagonal matrix as a sum of the diagonal part and a matrix of the small off-diagonal elements $\hat{H} = \hat{H}_d + \hat{V}$, $V \sim \mathcal{B} \ll 1$. The diagonal matrix $\hat{H}_d$ represents 'non-interacting' energy levels or localized eigenstates. The presence of the small off-diagonal matrix $\hat{V}$ leads to a weak 'interaction' between different localized states. In order to calculate correlation functions, one can perform an expansion in the number of interacting localized states [12]. The small parameter $\mathcal{B}$ is the control parameter of this procedure. This method was called [4–6] '*a virial expansion*' (VE) by analogy with the expansion of thermodynamic functions of a dilute system in powers of density with the mth virial coefficient being governed by collisions of $m+1$ particles.

These ideas were initially implemented in a semi-empirical real-space renormalization group approach which has been applied for critical systems with long-range interactions [12] and for a quantum Kepler problem [13]. The real-space renormalization group was also used to study the critical almost diagonal PLBRM [9]: the spectral correlations and the scaling properties of the eigenfunctions were investigated by considering a resonant interaction of two

energy levels. The renormalization group approach has however two serious disadvantages: (i) it does not allow for a rigorous control of omitted contributions; (ii) due to technical difficulties, it is almost impossible to go beyond the leading term, thus the resonant interaction of three and more levels remains inaccessible in this framework.

A rigorous counterpart of the renormalization group approach has been suggested in recent works [4–6]. It deals with a *regular* VE generated with the help of the Trotter formula (TVE) [14]. Based on the classification of perturbation series by a number of the interacting levels involved, TVE allows one to study the density of states and spectral correlations of the almost diagonal RMTs. The accuracy of the TVE is always controllable resulting in the rigorous perturbation theory. The second disadvantage of the renormalization group has been also partly overcome: the TVE allows us to go beyond the leading term by considering the interaction of two and three levels. In this way the first and the second virial coefficients have been calculated for a generic model of the almost diagonal RMTs. The general formulae have been applied [4–6] to the different models of almost diagonal RMTs, including critical PLBRMs [8], the unitary Moshe–Neuberger–Shapiro model [15] and the Rosenzweig–Porter model [16] in the regime of crossover.

The TVE involves a complicated combinatorial part of intermediate calculations. The combinatorial problem appeared there resembles a colouring problem of closed graph edges. If one considers the density of states [5], the colouring is similar to the well-studied problem of the graph theory [17] whose solution is known [18]. However, a study of spectral correlations requires to resolve much more complicated problem of simultaneous colouring of several graphs [4]. A complexity of the combinatorial calculations grows tremendously with increasing the number of the interacting energy levels. Therefore, the TVE can be used practically only for the calculation of the first and the second virial coefficients.

1.3. Virial expansion from the field-theoretical representation

In the present work, we formulate a *supersymmetric field-theoretical representation for the VE* of different correlation functions of the generic model (1). The method of the supersymmetry allows us to perform an averaging over RM ensemble for an arbitrary function $\mathcal{F}(|i-j|)$. The supersymmetric VE is controlled by a small parameter $\mathcal{B} \ll 1$. The virial coefficients are straightforwardly derived in a general form in terms of the integrals over supermatrices. It is important that no combinatorics appear in the intermediate field-theoretical calculations. In this framework, the interaction of m energy levels is described by an integral containing only m independent supermatrices associated with m different sites in the real space. In order to calculate the integrals over supermatrices explicitly we employ a parameterization introduced recently in [20].

The suggested supersymmetric field theory (SuSyFT) can be equally applied both to the spectral correlations and to the correlations of the eigenfunctions taken at different energies and in different space points. We emphasize that this approach is the unique analytic tool to describe the wavefunctions correlations for the almost diagonal RMTs. Neither the real-space renormalization group nor the TVE are capable to do this. We would also like to mention that SuSyFT might be useful for non-perturbative calculations as well, however this issue is beyond the scope of the present work.

The paper is organized as follows: we present the SuSyFT and give main definitions in section 2. The basic ideas and parameters of the VE are explained in section 3 with a reference to SuSyFT. A general integral expression for mth virial coefficient governed by the interaction of m supermatrices is presented in section 4. In this section, we also discuss in detail the validity of a saddle-point integration over massive degrees of freedom and the

applicability of the VE. We exemplify the integral calculations for the cases of 2- and 3-matrix interaction in section 5 and discuss advantages of the method and possible further applications in conclusions.

2. Main definitions

Let us introduce the retarded and advanced Green's functions

$$\hat{G}^{R/A}(E) = \frac{1}{E - \hat{H} \pm \imath 0}; \tag{2}$$

$\hat{H}$ is a Hermitian RM of large size $N \gg 1$. It has independent matrix elements and belongs to the Gaussian ensemble described by equation (1). Without loss of generality, we consider the case of GUE, $\beta = 2$. A generalization to other symmetry classes is straightforward. The Green's functions are an efficient tool to study different correlation functions. For example, the expression for the averaged density of states in terms of the Green's functions reads

$$\langle \rho(E) \rangle = \frac{1}{N} \sum_{n=1}^{N} \langle \delta(E - \epsilon_n) \rangle = \frac{1}{\pi N} \mathrm{Im} \langle \mathrm{Tr}(\hat{G}^A(E)) \rangle, \tag{3}$$

where ϵ_n are eigenvalues of the random matrix $\hat{H}$ and $\langle \cdots \rangle$ denotes averaging over the ensemble of random matrices. The inverse density of states taken at the band centre $E = 0$ governs the mean level spacing of RMT: $\Delta = 1/N\langle \rho(0) \rangle$. The mean level spacing of almost diagonal unitary RMT is [5]

$$\Delta|_{\mathcal{B} \ll 1} \simeq (\sqrt{\pi} + O(\mathcal{B}^2))N^{-1}.$$

The two-point correlation functions can be expressed by means of the quantity $\mathcal{G}_{pq}(\omega)$:

$$\mathcal{G}_{pq}(\omega) \equiv \frac{1}{\Delta} \int_{-\infty}^{\infty} \mathrm{d}E \hat{G}^R_{pp}(E + \omega/2) \hat{G}^A_{qq}(E - \omega/2). \tag{4}$$

For example, the expressions for the averaged two-level correlation function

$$R_2(\omega) \equiv N\Delta \int_{-\infty}^{\infty} \mathrm{d}E \langle\!\langle \rho(E + \omega/2)\rho(E - \omega/2) \rangle\!\rangle \tag{5}$$

is given by

$$R_2(\omega) = \frac{\Delta^2}{2\pi^2 N} \mathrm{Re} \sum_{p,q=1}^{N} \langle\!\langle \mathcal{G}_{pq}(\omega) \rangle\!\rangle, \tag{6}$$

where $\langle\!\langle ab \rangle\!\rangle \equiv \langle ab \rangle - \langle a \rangle \langle b \rangle$; while the averaged correlator of two eigenfunctions taken at different energies at different space points p and q

$$C_2(\omega, p, q) \equiv \Delta \int_{-\infty}^{\infty} \mathrm{d}E \left\langle\!\!\left\langle \sum_{m,n=1}^{N} \delta(E + \omega/2 - \epsilon_n)\delta(E - \omega/2 - \epsilon_m) \left|\psi_{\epsilon_n}(p)\right|^2 \left|\psi_{\epsilon_m}(q)\right|^2 \right\rangle\!\!\right\rangle \tag{7}$$

reads

$$C_2(\omega, p, q) = \frac{\Delta^2}{2\pi^2 N} \mathrm{Re} \langle\!\langle \mathcal{G}_{pq}(\omega) \rangle\!\rangle. \tag{8}$$

Thus, we have to calculate $\langle\!\langle \mathcal{G}_{pq}(\omega) \rangle\!\rangle$ to explore the two-point correlation functions. The ensemble averaging can be performed with the help of SuSyFT [11, 19]. To this end, we introduce N supervectors

$$\Psi(\alpha) = \begin{pmatrix} \Psi^R(\alpha) \\ \Psi^A(\alpha) \end{pmatrix}; \qquad \alpha = 1, 2, \ldots, N. \tag{9}$$

Here, $\Psi^{R/A}(\alpha) = (s^{R/A}(\alpha), \chi^{R/A}(\alpha))^T$ are supervectors in retarded-advanced sectors. They consist of commuting $s^{R/A}(\alpha)$ and anticommuting Grassmann variables $\chi^{R/A}(\alpha)$. The direct product of $\Psi(\alpha)$-vectors with the conjugated vectors constitutes N Q-matrices of the size 4×4. The block structure of the Q-matrices in the retarded-advanced notation is

$$Q_\alpha \equiv \Psi(\alpha)\otimes\bar{\Psi}(\alpha) = \begin{pmatrix} \Psi^R(\alpha)\otimes(\Psi^R(\alpha))^\dagger & \Psi^R(\alpha)\otimes K(\Psi^A(\alpha))^\dagger \\ \Psi^A(\alpha)\otimes(\Psi^R(\alpha))^\dagger & \Psi^A(\alpha)\otimes K(\Psi^A(\alpha))^\dagger \end{pmatrix}, \qquad K \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

The ensemble averaged $\mathcal{G}_{pq}$ can be written as follows:

$$\langle \mathcal{G}_{pq}(\omega)\rangle = \frac{(-1)^{N+1}}{\Delta}\int_{-\infty}^{\infty} \mathrm{d}E \int \mathcal{D}\{Q\}\mathcal{R}_p\mathcal{A}_q \left(\prod_{\alpha=1}^{N} \mathrm{e}^{S_0[Q_\alpha]}\right)\left(\prod_{n\neq m}^{N} \mathrm{e}^{S_p[Q_n,Q_m]}\right), \tag{11}$$

where $\mathcal{D}\{Q\} \equiv \prod_{\alpha=1}^{N}\mathcal{D}\{Q_\alpha\}$ is the measure of integration over the supermatrices Q_α (see appendix A for the details)[3]. The factors $\mathcal{R}_p$ and $\mathcal{A}_q$ break the supersymmetry between the commuting and anticommuting variables in the retarded/advanced sectors:

$$\mathcal{R}_p = (\chi^R(p))^*\chi^R(p), \qquad \mathcal{A}_q = (\chi^A(q))^*\chi^A(q). \tag{12}$$

In equation (11) we have separated out two parts of the action: S_0 corresponds to the diagonal part of RM and depends on a single supermatrix

$$S_0[Q_\alpha] = \mathrm{Str}\left\{-aQ_\alpha^2 + \imath\left(E + \frac{\Omega}{2}\Lambda\right)Q_\alpha\right\}, \qquad \Lambda \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{RA}, \tag{13}$$

$$a \equiv \frac{1}{2}\langle H_{\alpha\alpha}^2\rangle = \frac{1}{4}, \qquad \Omega \equiv \omega + \imath 0. \tag{14}$$

The second part of the action S_p is proportional to the variance of the off-diagonal elements and contains a product of two supermatrices

$$S_p[Q_k, Q_m] = -b_{km}\mathrm{Str}\{Q_kQ_m\}, \qquad k\neq m, \tag{15}$$

$$b_{km} \equiv \tfrac{1}{2}\langle |H_{km}|^2\rangle = \tfrac{1}{4}\mathcal{B}^2\mathcal{F}(|k-m|). \tag{16}$$

In other words, S_p describes an interaction of different Q-matrices. Note that equations (11), (13)–(16) are exact. We have used Q-matrices to compactify the notation but we could equally write the expression for $\langle \mathcal{G}_{pq}(\omega)\rangle$ in terms of the integrals over Ψ-vectors.

The standard σ-model derivation includes (i) introducing auxiliary supermatrices $Q_\alpha^{(\sigma)}$ that allow us to decouple the quartic in Ψ_α (i.e., quadratic in Q_α) terms by the Hubbard–Stratonovich transformation; (ii) Gaussian integration over the supervectors Ψ_α. After these two steps, one has to employ the saddle-point approximation for the integrals over $Q_\alpha^{(\sigma)}$. As we have already discussed in the introduction, this procedure is justified only for the large band width RMT, $\mathcal{B}\gg 1$. Here in contrast we will consider the case

$$\mathcal{B}\ll 1, \tag{17}$$

[3] Note that we have transformed the action written in terms of the supervectors Ψ with four commuting and four anticommuting variable to the action written in terms of the supermatrices Q with three commuting and four anticommuting variable. One phase of the commuting variables has been integrated out since the direct product (10) does not depend on it. This is possible for RMTs with uncorrelated matrix elements having zero mean value, cf equation (1).

where the standard σ-model fails. Therefore instead of the Hubbard–Stratonovich transformation with further standard steps, we will use the method of the virial expansion and employ a parameterization of the Q-matrices suggested in [20]. Details of the parameterization are given in appendix A. We emphasize that integration manifolds of the σ-model and of the VE *are quite different*: it is given by $\frac{U(1,1)}{U(1)\times U(1)} \times \frac{U(2)}{U(1)\times U(1)}$ for the former and by $\frac{U(1,1)}{U(1)\times U(1)} \times \mathbb{R}$ for the latter [20]. In particular, a saddle-point approximation in the SuSyFT suggested will lead to the linear constraint $\mathrm{Str}(Q) = 0$ for a large-scale theory (see section 4) whereas the saddle-point manifold of the standard diffusive σ-model requires the additional nonlinear constraint $(Q^{(\sigma)})^2 = 1$. Note however that the nonlinearity of SuSyFT follows already from the definition of the matrix $Q \equiv \Psi \otimes \bar{\Psi}$.

3. Basic concept of the virial expansion

The two-fold product of exponentials in equation (11) can be expanded in power series

$$\left(\prod_{n\neq m}^{N} \mathrm{e}^{S_p[Q_n,Q_m]}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m>n=1}^{N} (-2b_{m,n}\mathrm{Str}[Q_m Q_n])\right)^k \equiv \mathcal{V}^D + \sum_{m=2}^{\infty} \mathcal{V}^{(m)}; \qquad \mathcal{V}^D = 1. \tag{18}$$

We have rearranged this series by separating out terms $\mathcal{V}^{(m)}$ which contain a given number $m \geqslant 2$ of different Q-matrices. It is easy to show that

$$\mathcal{V}^{(2)} = \sum_{\alpha_1>\alpha_2=1}^{N} \mathcal{V}^{(2)}_{\alpha_1\alpha_2}, \qquad \mathcal{V}^{(2)}_{\alpha_1\alpha_2} \equiv \mathrm{e}^{-2b_{\alpha_1\alpha_2}\mathrm{Str}[Q_{\alpha_1}Q_{\alpha_2}]} - 1, \tag{19}$$

$$\begin{aligned} &\mathcal{V}^{(3)} = \sum_{\alpha_1>\alpha_2>\alpha_3=1}^{N} \mathcal{V}^{(3)}_{\alpha_1\alpha_2\alpha_3}, \\ &\mathcal{V}^{(3)}_{\alpha_1\alpha_2\alpha_3} \equiv \mathcal{V}^{(2)}_{\alpha_1\alpha_2}\mathcal{V}^{(2)}_{\alpha_1\alpha_3}\mathcal{V}^{(2)}_{\alpha_2\alpha_3} + \mathcal{V}^{(2)}_{\alpha_1\alpha_2}\mathcal{V}^{(2)}_{\alpha_1\alpha_3} + \mathcal{V}^{(2)}_{\alpha_1\alpha_2}\mathcal{V}^{(2)}_{\alpha_2\alpha_3} + \mathcal{V}^{(2)}_{\alpha_1\alpha_3}\mathcal{V}^{(2)}_{\alpha_2\alpha_3}, \end{aligned} \tag{20}$$

and so on for the higher terms. The function $\mathcal{V}^{(2)}_{\alpha_1\alpha_2}$ in VE is a counterpart of the Mayer's function used in the theory of imperfect gases [21].

This expansion can now be substituted into the expression for $\langle \mathcal{G}_{pq}(\omega)\rangle$:

$$\langle \mathcal{G}_{pq}(\omega)\rangle = \frac{(-1)^{N+1}}{\Delta}\int_{-\infty}^{\infty} \mathrm{d}E \int \mathcal{D}\{Q\}\mathcal{R}_p\mathcal{A}_q \left(\prod_{\alpha=1}^{N} \mathrm{e}^{S_0[Q_\alpha]}\right)(\mathcal{V}^D + \mathcal{V}^{(2)} + \mathcal{V}^{(3)} + \cdots). \tag{21}$$

The first term $\mathcal{V}^D = 1$ corresponds to the diagonal RMT with non-interacting localized eigenstates. This is the starting point for VE. The wavefunctions of the diagonal RMs are completely localized at different sites having no overlap with the other sites and the energy levels are uncorrelated:

$$\left\langle\!\left\langle \mathcal{G}^D_{p\neq q}\right\rangle\!\right\rangle = 0, \tag{22}$$

$$\left\langle\!\left\langle \mathcal{G}^D_{pp}\right\rangle\!\right\rangle = \frac{\pi}{\Delta}\left(\frac{2\imath}{\Omega} - \mathrm{e}^{-\frac{\omega^2}{2}}\int \mathrm{d}E\, \mathrm{e}^{-2E^2}\left(\mathrm{erfi}\left[E+\frac{\omega}{2}\right] - \imath\right)\left(\mathrm{erfi}\left[E-\frac{\omega}{2}\right] + \imath\right)\right); \tag{23}$$

$$\begin{aligned} &R_2^D\big|_{N\to\infty} = \mathrm{Re}\left(\frac{\Delta^2}{2\pi^2}\left\langle\!\left\langle \mathcal{G}^D_{pp}(\omega)\right\rangle\!\right\rangle\right) = \Delta(\delta(\omega) + O(1)) = \delta(s) + O(1/N); \\ &s \equiv \frac{\omega}{\Delta}, \qquad \mathrm{erfi}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t. \end{aligned} \tag{24}$$

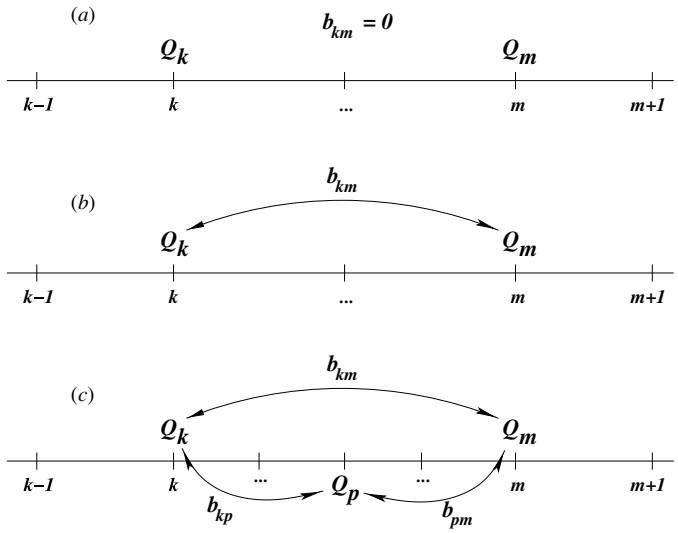

Figure 1. Scheme of the SuSyFT virial expansion: (*a*) supermatrices with different indices (associated with different sites in the real space) do not interact with each other in the case of the diagonal RMT possessing the localized wavefunctions; (*b*) pair interaction of the supermatrices via (small) off-diagonal elements of RMs, which governs the first virial coefficient $\tilde{\mathcal{G}}^{(2)}$, see equations (40) and (41); (*c*) the interaction of three supermatrices, which governs the second virial coefficient $\tilde{\mathcal{G}}^{(3)}$, see equations (40) and (41).

This is reflected by the structure of the first term: the Q-matrices are decoupled and all superintegrals factorize. We can say that the matrix Q_α is associated with the site α and the supermatrices at different sites do not interact with each other if we put $S_p = 0$ (see figure 1(*a*)).

Each term $\mathcal{V}^{(m)}$ is the sum of the exponentials containing m supermatrices. The supermatrices are linked by the small off-diagonal elements of RMs (see figures 1(*b*) and (*c*)). We refer to these links as 'the interaction of the supermatrices'. Obviously, they reflect the interaction between the localized eigenstates of the diagonal part of RMT.

Let us discuss general properties of the summands $\mathcal{V}^{(m)}_{\{\alpha\}}$, $\{\alpha\} \equiv \alpha_1, \alpha_2, \ldots, \alpha_m$. The set $\{\alpha\}$ must include external indices ($p \neq q$ and $p = q$ in the off-diagonal case and in the diagonal one, respectively); otherwise the contribution $\mathcal{V}^{(m)}_{\{\alpha\}}$ is cancelled by subtracting the decoupled term, see the definition of $\langle\langle \cdots \rangle\rangle$. The $N - m$ supermatrices, whose indices do not belong to the set $\{\alpha\}$, are included neither in $\mathcal{V}^{(m)}$ nor in the symmetry breaking factor $\mathcal{R}_p\mathcal{A}_q$ and, thus, are containing only in 'non-interacting' part of the action S_0. The integrals over these supermatrices are equal to unity due to the supersymmetry.

Let us introduce several definitions. We will call supermatrices Q_{β_1} and Q_{β_2} entering into the expression for $\mathcal{V}^{(m)}_{\{\alpha\}}$ *connected*, if and only if there is a sequence of indices $\{\gamma_1, \ldots, \gamma_n\}$ such that all elements $b_{\beta_1\gamma_1}$, $b_{\gamma_1\gamma_2}$, $\ldots$, $b_{\gamma_{n-1}\gamma_n}$, $b_{\gamma_n\beta_2}$ are contained in $\mathcal{V}^{(m)}_{\{\alpha\}}$. We refer to (i) a subset $\{\alpha'\} \subset \{\alpha\}$ as to *connected subset* if all independent matrices $Q_{\{\alpha'\}}$ are connected to each other; (ii) two non-intersecting subsets $\{\alpha^{(1)}\} \subset \{\alpha\}$, $\{\alpha^{(2)}\} \subset \{\alpha\}$, $\{\alpha^{(1)}\} \bigcap \{\alpha^{(2)}\} = \emptyset$ as to *disconnected subsets* if none of the matrix of the first subset $Q_{\{\alpha^{(1)}\}}$ is connected to any matrix of the second one $Q_{\{\alpha^{(2)}\}}$. If one or two external indices belong to a connected subset ($p \in \{\alpha'\}$ or $q \in \{\alpha'\}$ or $p, q \in \{\alpha'\}$) then all matrices of this subset are connected to the supermatrices with external indices. These connections break the supersymmetry for all supermatrices $Q_{\{\alpha'\}}$.

If $m = 2, 3$ then there are no disconnected subsets: all matrices entering $\mathcal{V}^{(m)}_{\{\alpha\}}$ are connected to each other and hence all of them are connected to the supermatrices with external indices and the supersymmetry is broken for all of them.

Starting from $m \geqslant 4$, the disconnected subsets may appear and there are $\mathcal{V}^{(m)}_{\{\alpha\}}$ containing supermatrices, which are not connected to any supermatrix with an external index. These are counterparts of the vacuum diagrams in the standard interaction representation [22] and they *do not contribute to the correlation functions*. For example, (i) if $\{\alpha^{(1,2)}\}$ are two disconnected subsets and $p, q \in \{\alpha^{(1)}\}$ then the supersymmetry is unbroken for all matrices $\{Q_{\{\alpha^{(2)}\}}\}$, the integrals over the supermatrices with unbroken supersymmetry yield trivial boundary contributions governed by $Q_{\{\alpha^{(2)}\}} = 0$ and the corresponding Mayer's functions are zero; (ii) if $\{\alpha^{(1,2)}\}$ are two disconnected subsets, $p \neq q$ and $p \in \{\alpha^{(1)}\}$, $q \in \{\alpha^{(2)}\}$ then the contribution $\mathcal{V}^{(m)}_{\{\alpha\}}$ is cancelled by subtracting the decoupled term (see the definition of $\langle\langle \cdots \rangle\rangle$).

To summarize the above discussion, each term $\mathcal{V}^{(m)}$ effectively consists only of irreducible parts where there are no disconnected matrices. The supersymmetry is broken for *all* m matrices (i) by the factor $\mathcal{RA}$ for the supermatrices $Q_{p,q}$ and (ii) by the inter-matrix links for the rest supermatrices. To formulate the VE, we focus just on these supermatrices $Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_m}$ with broken supersymmetry. Let us scale them by the constant[4] $\mathcal{B}$:

$$\tilde{Q}_{\alpha_j} = \mathcal{B} Q_{\alpha_j}, \qquad 1 \leqslant j \leqslant m. \tag{25}$$

After this scaling, the small parameter $\mathcal{B}$ is eliminated from $\mathcal{V}^{(m)}$ but appears in the single-matrix part S_0 in three different ratios as $1/\mathcal{B}^2$, $E/\mathcal{B}$ and $\Omega/\mathcal{B}$:

$$S_0\left[\frac{\tilde{Q}_{\alpha_j}}{\mathcal{B}}\right] = \mathrm{Str}\left\{-\frac{1}{4\mathcal{B}^2}\tilde{Q}^2_{\alpha_j} + \imath\left(\frac{E}{\mathcal{B}} + \frac{\Omega}{\mathcal{B}}\frac{\Lambda}{2}\right)\tilde{Q}_{\alpha_j}\right\}. \tag{26}$$

The first ratio is large, $1/\mathcal{B}^2 \gg 1$, and, therefore, the exponentials

$$\exp\left\{S_0\left[\frac{\tilde{Q}_{\alpha_j}}{\mathcal{B}}\right]\right\} \propto \exp\left\{-\frac{1}{4\mathcal{B}^2}\mathrm{Str}\big[\tilde{Q}^2_{\alpha_j}\big]\right\}$$

suppress the volume of integration over $\tilde{Q}_{\alpha_j}$ by constraint $\mathrm{Str}\big[\tilde{Q}^2_{\alpha_j}\big] < \mathcal{B}^2$. We can draw a conclusion that the larger the number m of independent Q-matrices the smaller is the contribution $\mathcal{G}^{(m)}_{pq}$ of m-matrix term $\mathcal{V}^{(m)}$ to the correlation function. This statement would be unquestionable if one integrates only over commuting variables. In the case of superintegrals it is more subtle and requires an additional discussion. An obvious counterexample to our estimate is the case of a partition function, for which the supersymmetry is unbroken, $\mathcal{R} = \mathcal{A} = 1$, and all superintegrals yield unity regardless of the apparent small phase volume of the integration. This happens due to the anomalous contributions [11] to the superintegrals. The irreducible terms $\mathcal{V}^{(m)}_{\{\alpha\}}$ can also contain the anomalous parts. The detailed analysis of the correlation function (see section 5.3) shows however that, due to the broken supersymmetry, the latter anomalies do not change the powers of the small parameter $\mathcal{B}$ and the following estimate holds true:

$$\frac{\mathcal{G}^{(m+1)}_{pq}}{\mathcal{G}^{(m)}_{pq}} \propto \mathcal{B}, \qquad m \geqslant 2. \tag{27}$$

[4] This scaling helps to single out two different parameters of the problem: the small parameter $\mathcal{B} \ll 1$ and an arbitrary ratio $\Omega/\mathcal{B}$. To this end we can scale by the constant either all variables of Ψ-vectors, equation (9), or only variables $\lambda_{R,A}$ of Q-matrices parameterized in accordance with appendix A. In both cases the integration measure is invariant due to the equal number of commuting and anticommuting variables in the former case and due to formula (A.7) in the latter one.

This justifies the supersymmetric virial expansion for the almost diagonal RMT: *the expansion of the correlation function in the number of the interacting supermatrices, equation* (21), *yields a regular perturbation theory in powers of* $\mathcal{B}$.

4. General expression for the virial coefficients

4.1. Saddle-point integration

Before presenting an expression for the arbitrary number of the interacting matrices, let us analyse the 2-matrix term $\langle\langle \mathcal{G}^{(2)}_{pq} \rangle\rangle$ in more detail. As we have already mentioned the supermatrices contained in $\mathcal{V}^{(2)}$ must be coupled to the factors $\mathcal{R}_p$ and $\mathcal{A}_{p,q}$. Therefore, we obtain the following expression for the off-diagonal $p \neq q$ and the diagonal $p = q$ parts of $\langle\langle \mathcal{G}^{(2)}_{pq} \rangle\rangle$:

$$\langle\langle \mathcal{G}^{(2)}_{p\neq q} \rangle\rangle = -\frac{2\pi}{\Delta} \langle \delta(\mathrm{Str}[Q_p + Q_q]) \mathcal{R}_p \mathcal{A}_q \mathcal{V}^{(2)}_{pq} \rangle_{Q_p, Q_q}, \tag{28}$$

$$\langle\langle \mathcal{G}^{(2)}_{pp} \rangle\rangle = -\frac{2\pi}{\Delta} \sum_{n=1, n\neq p}^{N} \langle \delta(\mathrm{Str}[Q_p + Q_n]) \mathcal{R}_p \mathcal{A}_p \mathcal{V}^{(2)}_{pn} \rangle_{Q_p, Q_n}. \tag{29}$$

We have introduced the averaging over the supermatrices:

$$\langle \cdots \rangle_{Q_\alpha} \equiv \int \mathcal{D}\{Q_\alpha\} (\cdots) \, \mathrm{e}^{\tilde{S}_0[Q_\alpha]}; \qquad \tilde{S}_0[Q_\alpha] \equiv S_0[Q_\alpha]\big|_{E=0} = -\frac{\mathrm{Str}\big[Q_\alpha^2\big]}{4} + \imath \frac{\Omega}{2} \mathrm{Str}[\Lambda Q_\alpha]. \tag{30}$$

The δ-function in equations (28) and (29) resulted from the integral over E, see equations (13) and (21).

Formulae for $\mathcal{G}^{(2)}_{pq}$ exactly describe the contribution of the two-level interaction to the correlation function $\mathcal{G}_{pq}$. However, $\mathcal{V}^{(2)}$ contains a product of two supermatrices, cf equation (19), which entangles the integration variables in a nontrivial way (see equation (A.10)). We will calculate the integral over the variables R_α (see equation (A.8)) in the saddle-point approximation. To explain this step, it is convenient to scale the supermatrices by $\sqrt{b_{p\alpha}}$

$$\bar{Q}_p = \sqrt{b_{p\alpha}} Q_p, \qquad \bar{Q}_\alpha = \sqrt{b_{p\alpha}} Q_\alpha;$$

consider the integration over the scaled variables $\bar{R}_\alpha$ and $\bar{S}_\alpha$ in equations (28) and (29) and perform the approximate integration over $\bar{R}$:

$$\int\!\!\int_{-\infty}^{\infty} \mathrm{d}\bar{R}_{p,\alpha} \int\!\!\int_{|\bar{R}_{p,\alpha}|}^{\infty} \mathrm{d}\bar{S}_{p,\alpha} \delta\left(\frac{\bar{R}_p + \bar{R}_\alpha}{\sqrt{b_{p\alpha}}}\right) \frac{\mathcal{R}_p \, \mathrm{e}^{-\frac{\bar{R}_p^2}{4b_{p\alpha}}}}{\bar{S}_p^2 - \bar{R}_p^2} \frac{\mathcal{A}_\alpha \, \mathrm{e}^{-\frac{\bar{R}_\alpha^2}{4b_{p\alpha}}}}{\bar{S}_\alpha^2 - \bar{R}_\alpha^2} \mathrm{e}^{\frac{\imath}{2} \frac{\Omega}{\sqrt{b_{p\alpha}}} (\bar{S}_p + \bar{S}_\alpha)} \mathcal{V}^{(2)}_{p,\alpha}(\bar{R}, \bar{S})$$

$$\simeq \sqrt{2\pi} b_{p\alpha} \left(\int\!\!\int_0^\infty \frac{\mathrm{d}\bar{S}_{p,\alpha}}{\bar{S}_p^2 \bar{S}_\alpha^2} \mathcal{R}_p \mathcal{A}_\alpha \Big|_{\bar{R}=0} \mathrm{e}^{\frac{\imath}{2} \frac{\Omega}{\sqrt{b_{p\alpha}}} (\bar{S}_p + \bar{S}_\alpha)} \mathcal{V}^{(2)}_{p,\alpha}(0, \bar{S}) + O(\sqrt{b_{p\alpha}}) + O\left(\frac{\bar{R}_t}{\bar{S}_t}\right) \right);$$

$$\bar{R}_\alpha = \mathrm{Str}[\bar{Q}_\alpha], \qquad \bar{S}_\alpha = \mathrm{Str}[\Lambda \bar{Q}_\alpha]. \tag{31}$$

Here $\alpha = n, p$ for the diagonal- and off-diagonal parts of $\mathcal{G}^{(2)}$, respectively. We have accounted for the δ-function in equations (28) and (29) and denoted the typical values of $\bar{S}$ and $\bar{R}$, at which the integrals converge, by $\bar{S}_t$ and $\bar{R}_t$ accordingly. The value of $\bar{R}_t$ is fixed by the Gaussian exponentials in equation (31): $\bar{R}_t \sim \sqrt{b_{p\alpha}}$. The integrals over $\bar{S}$ converge due to

the exponentials $e^{\frac{\iota}{2}\frac{\Omega}{\sqrt{b_{p\alpha}}}\bar{S}_{p,\alpha}}$, cf detailed calculations in section 5. Therefore, we can estimate the second characteristic scale as[5] $\bar{S}_t \sim \sqrt{b_{p\alpha}}/\omega$ and obtain $\bar{R}_t/\bar{S}_t \sim \omega$. The saddle-point integration over $\bar{R}$ makes sense only if the corrections in the right-hand part of equation (31) are small[6]:

$$\max\left\{\sqrt{b_{p\alpha}}, \frac{\bar{R}_t}{\bar{S}_t}\right\} \sim \max\{\sqrt{b_{p\alpha}}, \omega\} \ll 1. \tag{32}$$

Thus, we have to restrict ourselves to the region $\omega \ll 1$ where the density of states of the almost diagonal RMTs is close to the constant [5]. The ultrahigh frequencies $\omega \geqslant 1$ cannot be considered within the saddle-point integration over R and they are beyond the scope of the present paper. We can return to the unscaled matrices Q and arrive at the following equation for $\mathcal{G}^{(2)}$:

$$\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}\right\rangle\right\rangle = -\frac{(2\pi)^{3/2}}{\Delta}\left\langle \delta(\mathrm{Str}[Q_p])\delta(\mathrm{Str}[Q_q])\mathcal{R}_p\mathcal{A}_q\mathcal{V}^{(2)}_{pq}\right\rangle_{Q_p,Q_q} + \delta\mathcal{G}^{(2)}_{p\neq q}, \tag{33}$$

$$\left\langle\left\langle \mathcal{G}^{(2)}_{pp}\right\rangle\right\rangle = -\frac{(2\pi)^{3/2}}{\Delta}\sum_{n\neq p}^{N}\left\langle \delta(\mathrm{Str}[Q_p])\delta(\mathrm{Str}[Q_n])\mathcal{R}_p\mathcal{A}_p\mathcal{V}^{(2)}_{pn}\right\rangle_{Q_p,Q_n} + \delta\mathcal{G}^{(2)}_{pp}. \tag{34}$$

To calculate the leading terms, we have effectively replaced the exponentials $\exp(-R^2/4)$ in the integrand of (28)–(30) by the δ-functions of R:

$$\exp\left(-\tfrac{1}{4}\mathrm{Str}\left[Q_p^2 + Q_\alpha^2\right]\right)\delta(\mathrm{Str}[Q_p + Q_\alpha]) = \exp\left(-\tfrac{1}{4}\left(R_p^2 + R_\alpha^2\right)\right)\delta(R_p + R_\alpha)$$
$$\to \sqrt{2\pi}\delta(R_\alpha)\delta(R_p) = \sqrt{2\pi}\delta(\mathrm{Str}[Q_\alpha])\delta(\mathrm{Str}[Q_p]). \tag{35}$$

Equations (31) and (32) implies in the case of the off-diagonal correlation function:

$$\frac{\delta\mathcal{G}^{(2)}_{p\neq q}}{\mathcal{G}^{(2)}_{p\neq q}} \sim \max\{\mathcal{B}\sqrt{\mathcal{F}(|p-q|)}, \omega\}. \tag{36}$$

A similar estimate for the diagonal correlation function is more subtle since the derivation of $\mathcal{G}^{(2)}_{pp}$ involves the summation over the auxiliary index n. Let us assume that this sum in the leading part of $\mathcal{G}^{(2)}_{pp}$ as well as in the correction $\delta\mathcal{G}^{(2)}_{pp}$ converges at a characteristic scale $|n-p| \sim X_c$, then we can expect that

$$\frac{\delta\mathcal{G}^{(2)}_{pp}}{\mathcal{G}^{(2)}_{pp}} \sim \max\{\mathcal{B}\sqrt{\mathcal{F}(X_c)}, \omega\}. \tag{37}$$

The value of X_c is, of course, model dependent and varies for different RMTs.

The generalization of equations (33) and (34) for an arbitrary number m of the interacting Q-matrices reads

$$\left\langle\left\langle \mathcal{G}^{(m)}_{p\neq q}\right\rangle\right\rangle \simeq \frac{2\pi}{\Delta}\frac{(-2\sqrt{\pi})^{m-1}}{\sqrt{m}}\sum_{\{\alpha_j\neq p,q\}}^{N}\left\langle \delta(\mathrm{Str}[Q_p])\delta(\mathrm{Str}[Q_q])\mathcal{R}_p\mathcal{A}_q\right.$$
$$\left.\times\left(\prod_j \delta\left(\mathrm{Str}\left[Q_{\alpha_j}\right]\right)\right)\mathcal{V}^{(m)}_{pq\alpha_1\alpha_2\ldots\alpha_{m-2}}\right\rangle_{Q_p,Q_q\,Q_{\alpha_1}\,Q_{\alpha_2}\ldots Q_{\alpha_{m-2}}} \qquad 1 \leqslant j \leqslant m-2; \tag{38}$$

[5] This is obvious if one deforms the integration contour for the S-variable from the real axis $S \in [0, +\infty]$ to the imaginary one $S \in [0, +\iota\infty]$.

[6] Small corrections to the saddle-point integration in equation (31) have been obtained by a rough upper estimate. More detailed analysis shows that they can be parametrically smaller. For example, the corrections $O(\sqrt{b_{p\alpha}})$ cancel out in the leading term of the virial expansion for the level compressibility and only the smaller ones $O(b_{p\alpha})$ remain. In such cases, validity of the saddle-point integration becomes even broader than the range described by equation (32).

$$\left\langle\!\left\langle \mathcal{G}_{pp}^{(m)} \right\rangle\!\right\rangle \simeq \frac{2\pi}{\Delta}\frac{(-2\sqrt{\pi})^{m-1}}{\sqrt{m}} \sum_{\{\alpha_j \neq p\}}^{N} \left\langle \delta(\mathrm{Str}[Q_p])\mathcal{R}_p\mathcal{A}_p \right.$$
$$\left. \times \left(\prod_j \delta\big(\mathrm{Str}\big[Q_{\alpha_j}\big]\big)\right)\mathcal{V}^{(m)}_{pq\alpha_1\alpha_2\ldots\alpha_{m-1}} \right\rangle_{Q_p, Q_q Q_{\alpha_1} Q_{\alpha_2}\cdots Q_{\alpha_{m-1}}} \qquad 1 \leqslant j \leqslant m-1. \tag{39}$$

The summation is performed over ordered indices $1 \leqslant \alpha_1 < \alpha_2 < \alpha_3 < \cdots \leqslant N$ excluding the external fixed indices p and q. It is easy to show (see section 5.3) that these expressions have the following functional dependence on parameters Ω and $\mathcal{B}$:

$$\left\langle\!\left\langle \mathcal{G}^{(m)} \right\rangle\!\right\rangle = \mathcal{B}^{m-2}\bar{\mathcal{G}}^{(m)}(\Omega/\mathcal{B}), \tag{40}$$

with function $\bar{\mathcal{G}}^{(m)}$ depending only on the ratio $\Omega/\mathcal{B}$. Thus one can write VE of the correlation function as a functional series in powers of $\mathcal{B}$:

$$\left\langle\!\left\langle \mathcal{G}_{pq}(\mathcal{B}, \Omega/\mathcal{B}) \right\rangle\!\right\rangle \simeq \bar{\mathcal{G}}^{D}_{pq} + \sum_{m\geqslant 2} \mathcal{B}^{m-2}\bar{\mathcal{G}}^{(m)}_{pq}(\Omega/\mathcal{B}). \tag{41}$$

The functions $\bar{\mathcal{G}}^{(m)}_{pq}$ are the *virial coefficients*. Each coefficient $\bar{\mathcal{G}}^{(m)}(\Omega/\mathcal{B})$ is governed by the interaction of m supermatrices corresponding to the interaction of m localized states. The first term $\bar{\mathcal{G}}^{D}$ is related to uncorrelated statistics of the diagonal part of the almost diagonal RMTs.

By analogy with the estimates (36) and (37), equations (38) and (39) describe an arbitrary virial coefficient $\bar{\mathcal{G}}^{(m)}$ with the following accuracy:

$$\frac{\delta\mathcal{G}^{(m)}_{p\neq q}}{\mathcal{G}^{(m)}_{p\neq q}} \sim \max\{\mathcal{B}\sqrt{\mathcal{F}(|p-q|)}, \omega\}, \qquad \frac{\delta\mathcal{G}^{(m)}_{pp}}{\mathcal{G}^{(m)}_{pp}} \sim \max\{\mathcal{B}\sqrt{\mathcal{F}(X_c)}, \omega\}. \tag{42}$$

It is convenient to represent the corrections schematically as a sum of two terms:

$$\delta\mathcal{G}^{(m)}_{pq} = \delta_\omega\mathcal{G}^{(m)}_{pq} + \delta_{\mathcal{B}}\mathcal{G}^{(m)}_{pq}$$

where $\delta_\omega\mathcal{G}^{(m)}_{pq}\big/\mathcal{G}^{(m)}_{pq} \sim \omega$, $\delta_{\mathcal{B}}\mathcal{G}^{(m)}_{p\neq q}\big/\mathcal{G}^{(m)}_{p\neq q} \sim \mathcal{B}\sqrt{\mathcal{F}(|p-q|)}$ and $\delta_{\mathcal{B}}\mathcal{G}^{(m)}_{pp}\big/\mathcal{G}^{(m)}_{pp} \sim \mathcal{B}\sqrt{\mathcal{F}(X_c)}$.

4.2. Validity of the virial expansion and the large-scale limit

All details of the evaluation of $\mathcal{G}^{(2,3)}$ are presented in the next section. Here, we would like to discuss the validity and the applicability of VE (38)–(41).

Firstly we note that VE (41) is a functional series. Its successive terms decrease with increasing the number of the interacting supermatrices only if the absolute value of the virial coefficients is bounded for the arbitrary ratio $\Omega/\mathcal{B}$. This condition determines a convergence of VE but it cannot be checked until the RMT model is specified. In particular, this condition is violated for RMTs with almost ergodic wavefunctions.

Secondly we should recall that, calculating $\bar{\mathcal{G}}^{(m)}(\Omega/\mathcal{B})$ by the saddle-point approximation in R-variables, we have neglected corrections $\delta_\omega\mathcal{G}^{(m)}_{pq}$ which are of the order of $O(\omega)$. This means that for a given ω the summation over m in VE described by equations (38)–(41) must be stopped at

$$m_{\max} \sim 1 + \log(\omega)/\log(\mathcal{B}).$$

For instance, if $\mathcal{B} \leqslant \omega \ll 1$ then $\delta_\omega\mathcal{G}^{(2)} \geqslant \mathcal{G}^{(3)}$ and one may take into account only the interaction of two supermatrices neglecting all higher terms. The next term of VE governed by the interaction of three supermatrices may be taken into consideration only for the smaller energy $\omega \ll \mathcal{B}$ when $\delta_\omega\mathcal{G}^{(2)} \ll \mathcal{G}^{(3)}$, etc. On the other hand, the neglected dependence of the

virial coefficients on ω often results from the energy dependence of the density of state and does not influence universal properties of the correlation functions.

Finally let us compare $\delta_{\mathcal{B}}\mathcal{G}^{(m)}$ with the successive term of VE $\mathcal{G}^{(m+1)}$. For the sake of simplicity we compare $\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}$ with $\mathcal{G}^{(3)}_{p\neq q}$ though the same analysis applies to the diagonal virial coefficients with $p = q$ and for the arbitrary m. Without loss of generality we put $\mathcal{F}(1) = 1$. If $|p-q| \sim 1$ then $\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}/\mathcal{G}^{(2)}_{p\neq q} \sim \mathcal{B}$ and there is no way to get the scale separation: $\delta_{\mathcal{B}}\mathcal{G}^{(2)}$ and $\mathcal{G}^{(3)}$ are of the same order and, again, one may consider only the two matrix interaction regardless of the energy smallness.

Now, we will show that $\delta_{\mathcal{B}}\mathcal{G}^{(2)}$ is parametrically smaller than the higher terms of VE in *the large-scale limit*. The large-scale limit means that we consider only those correlation functions which are not sensitive to the details of $\mathcal{F}$ at small distances and governed by the behaviour of this function at large distances X_c, at which $\mathcal{F}$ is sufficiently small $\mathcal{F}(X_c) \ll 1$. It means, in particular, that we assume: (i) $|p-q| \gtrsim X_c$ in the case of $\mathcal{G}^{(2)}_{p\neq q}$; (ii) the main contribution to the sum over the auxiliary index α_1 originates from $|\alpha_1 - p| \gtrsim X_c$ and $|\alpha_1 - q| \gtrsim X_c$ in the case of $\mathcal{G}^{(3)}_{p\neq q}$. We recall that the sum over α_1 is due to presence of the third supermatrix, see equation (38). Although the second assumption is not applicable for arbitrary function $\mathcal{F}$, it allows us to study a wide class of almost diagonal RMTs. We arrive at the following estimate in the framework of the large-scale limit:

$$\frac{\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}}{\mathcal{G}^{(2)}_{p\neq q}} \sim \mathcal{B}\mathcal{F}(|p-q|) \leqslant \mathcal{B}\mathcal{F}(X_c) \ll \mathcal{B}.$$

The ratio $\mathcal{G}^{(3)}_{p\neq q}/\mathcal{G}^{(2)}_{p\neq q}$ requires a separate consideration: the presence of the third supermatrix in the expression for $\mathcal{G}^{(3)}_{p\neq q}$ results in the additional factors $\mathcal{B}\sqrt{\mathcal{F}(|p-\alpha_1|)}$ or $\mathcal{B}\sqrt{\mathcal{F}(|q-\alpha|)}$ and simultaneously requires an additional summation over the index α_1, see detailed calculations in section 5.2. This summation is crucial: we have assumed that it converges at $|p-m| \sim |q-m| \sim X_c$, but the large phase volume of summation can *compensate the smallness* of $\sqrt{\mathcal{F}(X_c)}$, cf [6]. If this is the case we obtain

$$\frac{\mathcal{G}^{(3)}_{p\neq q}}{\mathcal{G}^{(2)}_{p\neq q}} \sim \mathcal{B} \quad \Rightarrow \quad \frac{\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}}{\mathcal{G}^{(3)}_{p\neq q}} \sim \sqrt{\mathcal{F}(X_c)} \ll 1.$$

Thus, if the assumptions of the large-scale limit hold true then $\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}$ is parametrically smaller than the next term of the virial expansion. In particular if the characteristic scale X_c depends on N and the function $\mathcal{F}$ decreases in such a way that $\lim_{N\to\infty}\sqrt{\mathcal{F}(X_c)} = 0$ then the ratio $\delta_{\mathcal{B}}\mathcal{G}^{(2)}_{p\neq q}/\mathcal{G}^{(2,3)}_{p\neq q}$ asymptotically goes to zero. This scale separation justifies VE in the large-scale limit in many cases. One can check, for example, that it is correct for the spectral statistics of the critical almost diagonal PLBRMs, where $X_c \propto \mathcal{B}N$ and the relevant energy range reads $\omega \leqslant \mathcal{B}\Delta$ [4].

We would like to mention that the large-scale limit considered here is analogous to the diffusive approximation of the standard σ-model. In the latter approximation, the spatial scales large compared to the mean free path are assumed to be the only relevant ones. In the same time, the saddle-point approximation justified in the large-scale limit in our approach results in the linear constraint on the Q-matrix

$$\mathrm{Str}[Q] = 0, \tag{43}$$

while the saddle-point manifold of the standard diffusive σ-model is defined by the additional nonlinear constraint $(Q^{(\sigma)})^2 = 1$.

4.3. Correlation function at the band centre

The expressions for the correlation functions given by equations (38) and (39) were derived after integration over E. However in some applications it is more convenient to consider a correlation function at a fixed energy. The aim of this section is to briefly discuss the correlation functions at the band centre $E = 0$.

We define the two-point correlation function at the band centre as

$$\bar{\mathcal{G}}_{pq}(\omega, E = 0) \equiv \hat{G}^R_{pp}(\omega/2)\hat{G}^A_{qq}(-\omega/2). \tag{44}$$

Repeating all the steps leading to the results equation (38), (39), we obtain

$$\left\langle\left\langle \bar{\mathcal{G}}^{(m)}_{pq}(\omega, E = 0)\right\rangle\right\rangle \simeq \frac{\sqrt{m}}{N}\left\langle\left\langle \mathcal{G}^{(m)}_{pq}(\omega)\right\rangle\right\rangle. \tag{45}$$

The ratio $\sqrt{m}/N$ can be referred to as 'the unfolding factor'[7].

Let us also note that an average of the product of two retarded (advanced) Green's functions can be neglected again for the following reason. If we look at the parameterization of Q in this case (appendix B), then we note that variables R and S change their roles. For this reason the large-scale approximation $\mathrm{Str}[Q] \approx 0$ implies now $\lambda_R, \lambda_{R'} \approx 0$. Thus, the volume of the integration in this case becomes parametrically small at $\mathcal{B} \ll 1$ and $\omega \ll 1$.

Below, we will analyse only the correlation functions averaged over E.

5. The cases of 2- and 3-matrix approximation

In this section, we present detailed calculation of the contributions $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(3)}$ governed by the interaction of two and three supermatrices, respectively. It is more convenient to expand $\mathcal{V}^{(2,3)}$ in the series of powers of Q-matrices and then integrate over Q term by term. This step is not essential for the approximation of two interacting matrices which can be worked out directly from equations (33) and (34). However, it is more convenient for the 3-matrix approximation, since it allows one to unify the calculations for the different number of the interacting supermatrices and to avoid an explicit derivation of anomalous terms in the superintegrals. The disadvantage of this route is that the series obtained converge only asymptotically and one has to Fourier transform them in order to analyse the answer in the time domain [4].

5.1. The case of two interacting supermatrices

We start with calculating $\mathcal{G}^{(2)}$ using approximate formulae (33) and (34). The power series for $\mathcal{V}^{(2)}$ reads

$$\mathcal{V}^{(2)}_{pn} = \sum_{k=1}^{\infty} \frac{(-2b_{pn}\mathrm{Str}[Q_p Q_n])^k}{k!}. \tag{46}$$

We use the phase ϕ and two non-compact variables R and S to parameterize the boson–boson sector of each supermatrix (see the corresponding definitions in appendix A). The integration measure in equations (33) and (34) takes the form

$$\int \mathcal{D}\{Q\}\delta(\mathrm{Str}[Q])(\cdots) \to 2\int_{-\infty}^{\infty} \mathrm{d}R\, \delta(R) \int_0^{\infty} \frac{\mathrm{d}S}{S^2}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\{\eta^*_R \eta_R \eta^*_A \eta_A\}(\cdots) \tag{47}$$

[7] Note that the unfolding factor $\sqrt{2}$ has been calculated in [4] for the level compressibility in the case $m = 2$ and it agrees with equation (45). However in the subsequent paper [6] the same unfolding factor has been erroneously used for $m = 3$. The correct unfolding factor is $\sqrt{3}$, cf equation (45).

and the expressions for $\mathcal{R}_p\mathcal{A}_{q,p}$ read

$$\mathcal{R}_p\mathcal{A}_q|_{R_{p,q}=0} = \tfrac{1}{4}S_pS_q(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q; \tag{48}$$

$$\mathcal{R}_p\mathcal{A}_p|_{R_p=0} = \tfrac{1}{4}S_p^2(\eta_R^*\eta_R\eta_A^*\eta_A)_p. \tag{49}$$

Integrating over R-variables we obtain

$$\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}\right\rangle\right\rangle \simeq -\frac{(2\pi)^{3/2}}{\Delta}\sum_{k=1}^{\infty}\int\!\!\int_0^{\infty}\frac{\mathrm{d}S_{p,q}}{S_pS_q}\,\mathrm{e}^{\imath\frac{\Omega}{2}(S_p+S_q)}\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q}$$
$$\times(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q\int\!\!\int_0^{2\pi}\frac{\mathrm{d}\phi_{p,q}}{(2\pi)^2}\frac{(-2b_{pq}\mathcal{S}_{pq})^k}{k!}, \tag{50}$$

$$\left\langle\left\langle \mathcal{G}^{(2)}_{pp}\right\rangle\right\rangle \simeq -\frac{(2\pi)^{3/2}}{\Delta}\sum_{n\neq p}^{N}\sum_{k=1}^{\infty}\int\!\!\int_0^{\infty}\frac{\mathrm{d}S_{p,n}}{S_n^2}\,\mathrm{e}^{\imath\frac{\Omega}{2}(S_p+S_n)}\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,n}$$
$$\times(\eta_R^*\eta_R\eta_A^*\eta_A)_p\int\!\!\int_0^{2\pi}\frac{\mathrm{d}\phi_{p,n}}{(2\pi)^2}\frac{(-2b_{pn}\mathcal{S}_{pn})^k}{k!}. \tag{51}$$

The expression for $\mathcal{S}_{pq} \equiv \mathrm{Str}[Q_pQ_q]|_{R_{p,q}=0} \propto S_pS_q$ is given in appendix D, equations (D.3) and (D.5). The integrals of $\mathcal{S}_{pq}^k$ over the phases are calculated in the same appendix, equations (D.2) and (D.4). The integrals over the S-variables are regularized at the upper limit by the imaginary part of Ω and converge at the lower limit for all k in the case of $\mathcal{G}^{(2)}_{p\neq q}$ and for $k \geqslant 2$ in the case of $\mathcal{G}^{(2)}_{pp}$. The term with $k = 1$ in the diagonal part $\mathcal{G}^{(2)}_{pp}$ is special: it is governed by an anomaly, i.e., an uncertainty $0\times\infty$ with zero resulting from the integrals over the Grassmann variables $(\eta_R^*\eta_R\eta_A^*\eta_A)_n$ and infinity due to the divergence at the lower limit of integration over the commuting variable S_n. This uncertainty can be resolved either in a standard way [11] or, equally, one can calculate the integrals in the diagonal part for $k \geqslant 2$ and then perform an analytic continuation for $k = 1$. The result of the integration over all variables can be written as follows:

$$\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}\right\rangle\right\rangle \simeq \frac{(2\pi)^{3/2}}{\Delta}\sum_{k=1}^{\infty}\left(\frac{2b_{pq}}{\Omega^2}\right)^k\frac{\Gamma(2k-1)}{\Gamma(k)}(k-1), \tag{52}$$

$$\left\langle\left\langle \mathcal{G}^{(2)}_{pp}\right\rangle\right\rangle \simeq \frac{(2\pi)^{3/2}}{\Delta}\sum_{n\neq p}^{N}\sum_{k=1}^{\infty}\left(\frac{2b_{pn}}{\Omega^2}\right)^k\frac{\Gamma(2k-1)}{\Gamma(k)}k. \tag{53}$$

The correlation functions R_2 and C_2 can be calculated from the real part of $\mathcal{G}_{pq}$ (see equations (6) and (8)). Taking the real part by substituting ω instead of Ω in equations (52) and (53) one obtains the asymptotic series in the energy representation. However, it is more convenient to consider the time representation by performing the Fourier transform of the real part

$$\mathcal{G}_{pq}(t) = \frac{1}{2\Delta}\int \mathrm{d}\omega\,\mathrm{e}^{-\imath\omega t}(\mathcal{G}_{pq}(\omega) + \mathrm{c.c.}),$$

obtaining

$$\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}(t)\right\rangle\right\rangle \simeq \frac{\pi(2\pi)^{3/2}}{\Delta^2|t|}\sum_{k=1}^{\infty}\frac{(-2b_{pq}t^2)^k}{(k-1)!}\frac{k-1}{2k-1}, \tag{54}$$

$$\left\langle\left\langle \mathcal{G}^{(2)}_{pp}(t)\right\rangle\right\rangle \simeq \frac{\pi(2\pi)^{3/2}}{\Delta^2|t|}\sum_{n\neq p}^{N}\sum_{k=1}^{\infty}\frac{(-2b_{pn}t^2)^k}{(k-1)!}\frac{k}{2k-1}. \tag{55}$$

Now the summation over k can be done explicitly

$$\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}(t)\right\rangle\right\rangle \simeq -\sqrt{2}\frac{\pi^{5/2}}{\Delta^2}\sqrt{2b_{pq}}\left[\sqrt{2b_{pq}}|t|\,\mathrm{e}^{-2b_{pq}t^2}-\frac{\sqrt{\pi}}{2}\mathrm{erf}(\sqrt{2b_{pq}}|t|)\right], \tag{56}$$

$$\left\langle\left\langle \mathcal{G}^{(2)}_{pp}(t)\right\rangle\right\rangle \simeq -\sqrt{2}\frac{\pi^{5/2}}{\Delta^2}\sum_{n\neq p}^{N}\sqrt{2b_{pn}}\left[\sqrt{2b_{pn}}|t|\,\mathrm{e}^{-2b_{pn}t^2}+\frac{\sqrt{\pi}}{2}\mathrm{erf}(\sqrt{2b_{pn}}|t|)\right]. \tag{57}$$

Here $\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t$. Our theory can be verified by comparison with the results of TVE. To this end we calculate the form factor, which is the Fourier transform of the two-level correlation function R_2:

$$K(t)=\frac{\Delta^2}{2\pi^2 N}\,\mathrm{Re}\sum_{p,q=1}^{N}\langle\langle \mathcal{G}_{pq}(t)\rangle\rangle, \tag{58}$$

and insert in this formula equations (56) and (57). This gives the form factor in the approximation of two interacting levels:

$$K^{(2)}(t)\simeq -\frac{\sqrt{2\pi}}{N|t|}\sum_{p,q=1}^{N}x(|p-q|)\,\mathrm{e}^{-x(|p-q|)}\simeq\Big|_{N\gg 1}-2\frac{\sqrt{2\pi}}{|t|}\sum_{m=1}^{N}x(m)\,\mathrm{e}^{-x(m)}, \tag{59}$$

$$x(|p-q|)\equiv 2b_{pq}t^2=\frac{1}{2}(\mathcal{B}t)^2\mathcal{F}(|p-q|),$$

which coincides with the expression for $K^{(2)}(t)$ obtained by TVE [4].

This comparison of TVE and the theory based on SuSyFT clearly demonstrates that SuSyFT is capable to give much more detailed information on the correlation functions. Namely, TVE deals with the form factor which is an integral quantity obtained after the summation of diagonal and off-diagonal parts of the correlation function $\mathcal{G}$ over all spatial coordinates, while the correlation function $\mathcal{G}$ at given spatial points can be derived only from SuSyFT.

We can now return from equations (56) and (57) written in the time domain to the energy representation of $\mathrm{Re}[\mathcal{G}^{(2)}]$:

$$\mathrm{Re}\left\langle\left\langle \mathcal{G}^{(2)}_{p\neq q}(\omega)\right\rangle\right\rangle \simeq -\frac{\pi^{3/2}}{\sqrt{2}\Delta}\left[1-\frac{\sqrt{\pi}}{2}\,\mathrm{e}^{-\frac{\omega^2}{8b_{pq}}}\left(\frac{\omega}{\sqrt{2b_{pq}}}-\frac{\sqrt{8b_{pq}}}{\omega}\right)\mathrm{erfi}\left(\frac{\omega}{\sqrt{8b_{pq}}}\right)\right], \tag{60}$$

$$\mathrm{Re}\left\langle\left\langle \mathcal{G}^{(2)}_{pp}(\omega)\right\rangle\right\rangle \simeq -\frac{\pi^{3/2}}{\sqrt{2}\Delta}\sum_{n\neq p}^{N}\left[1-\frac{\sqrt{\pi}}{2}\,\mathrm{e}^{-\frac{\omega^2}{8b_{pn}}}\left(\frac{\omega}{\sqrt{2b_{pn}}}+\frac{\sqrt{8b_{pn}}}{\omega}\right)\mathrm{erfi}\left(\frac{\omega}{\sqrt{8b_{pn}}}\right)\right]. \tag{61}$$

The power series equations (52) and (53) are asymptotic expansion of these formulae [23]. Note that the summands on the right-hand side of equation (61) are peaked around the value

$$\frac{\omega}{\sqrt{8b_{pn}}}\equiv\frac{\omega}{2\mathcal{B}\sqrt{\mathcal{F}(|p-n|)}}\sim 1, \tag{62}$$

see figure 2. Thus, we can find the characteristic spatial scale X_c, which yields the main contribution to the sum over n and determines $\mathcal{G}^{(2)}_{pp}(s)$, from the following estimate:

$$\mathcal{F}(X_c)\sim\left(\frac{\omega}{\mathcal{B}}\right)^2. \tag{63}$$

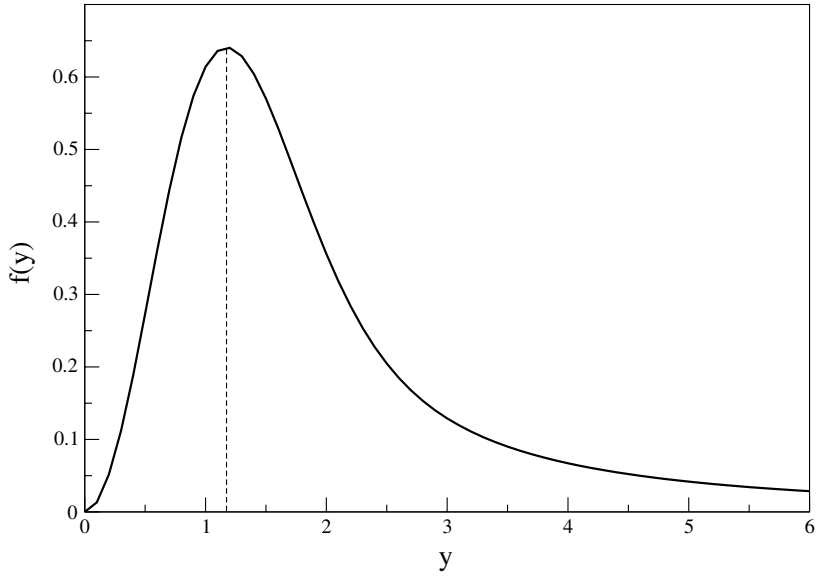

Figure 2. Summand in the right hand side of equation (61): $f(y) = -[1-\frac{\sqrt{\pi}}{2}e^{-y^2}(2y+\frac{1}{y})\mathrm{erfi}(y)]$. Here the argument y denotes the parameter $\omega/\sqrt{8b_{pn}}$.

Estimate (63) ensures the validity of the large-scale limit for $\omega \ll \mathcal{B} \ll 1$ at the level of two-matrix approximation. Indeed if $\omega \ll \mathcal{B}$ then $\mathcal{F}(X_c) \ll 1$, hence $X_c \gg 1$, i.e., the diagonal correlator $\mathcal{G}^{(2)}_{pp}$ is governed by the large distances, and the correction $\delta_{\mathcal{B}}\mathcal{G}^{(2)}$ to the saddle-point integration is expected to be smaller than the higher terms of the VE. In contrast, in the range $\mathcal{B} \leqslant \omega \ll 1$ the characteristic scale is small, $X_c \to 1$ and the higher terms of the VE can be of the same order as the omitted correction $\delta_{\mathcal{B}}\mathcal{G}^{(2)}$, see section 4.2. This means that we cannot use the saddle-point integration to go beyond the two-matrix approximation in the case $\mathcal{B} \leqslant \omega \ll 1$.

5.2. *The case of three interacting supermatrices*

The calculations of $\mathcal{G}^{(3)}$ based on equations (38) and (39) are very similar to those described in the preceding section for $\mathcal{G}^{(2)}$. The power series for $\mathcal{V}^{(3)}$ read

$$\mathcal{V}^{(3)}_{pmn} = \left\{ \sum_{k_{1,2,3}=1}^{\infty} + \sum_{k_{1,2}=1}^{\infty}\Big|_{k_3=0} + \sum_{k_{1,3}=1}^{\infty}\Big|_{k_2=0} + \sum_{k_{2,3}=1}^{\infty}\Big|_{k_1=0} \right\} \times \frac{(-2b_{pn}\mathrm{Str}[Q_pQ_n])^{k_1}}{k_1!}\frac{(-2b_{pm}\mathrm{Str}[Q_pQ_m])^{k_2}}{k_2!}\frac{(-2b_{mn}\mathrm{Str}[Q_mQ_n])^{k_3}}{k_3!}. \quad (64)$$

We insert this series into equations (38) and (39) and integrate over R-variables obtaining

$$\left\langle\left\langle \mathcal{G}^{(3)}_{p\neq q}\right\rangle\right\rangle \simeq \frac{(4\pi)^2}{\sqrt{3}\Delta}\sum_{\{m\neq p,q\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}\iiint_0^{\infty}\frac{\mathrm{d}S_{p,q,m}}{S_pS_qS_m^2}\,e^{\imath\frac{\Omega}{2}(S_p+S_q+S_m)}\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m} \times(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q\iiint_0^{2\pi}\frac{\mathrm{d}\phi_{p,q,m}}{(2\pi)^3}\frac{(-2b_{pq}\mathcal{S}_{pq})^{k_1}}{k_1!} \times\frac{(-2b_{pm}\mathcal{S}_{pm})^{k_2}}{k_2!}\frac{(-2b_{qm}\mathcal{S}_{qm})^{k_3}}{k_3!}, \quad (65)$$

$$\left\langle\left\langle \mathcal{G}^{(3)}_{pp}\right\rangle\right\rangle \simeq \frac{(4\pi)^2}{\sqrt{3}\Delta}\sum_{\{m,n\neq p\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}\iiint_0^\infty \frac{\mathrm{d}S_{p,m,n}}{S_m^2 S_n^2}\,\mathrm{e}^{\imath\frac{\Omega}{2}(S_p+S_m+S_n)}\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,m,n}$$
$$\times(\eta_R^*\eta_R\eta_A^*\eta_A)_p\iiint_0^{2\pi}\frac{\mathrm{d}\phi_{p,m,n}}{(2\pi)^2}\frac{(-2b_{pq}\mathcal{S}_{pq})^{k_1}}{k_1!}\frac{(-2b_{pm}\mathcal{S}_{pm})^{k_2}}{k_2!}$$
$$\times\frac{(-2b_{qm}\mathcal{S}_{qm})^{k_3}}{k_3!};\qquad m>n. \tag{66}$$

We use the same trick with the analytical continuation from $k_j \geqslant 2$ to $k_j = 1$ to handle the anomalous terms. Note that we have put zero in the lower limit for the sum over $k_{1,2,3}$ and combined all four contribution in equation (64) together. This is possible since all terms with either $k_1 = k_2 = 0$ or $k_1 = k_3 = 0$ or $k_2 = k_3 = 0$ are equal to zero after the analytical continuation (see the results for $\mathcal{G}^{(3)}$ below). The integrals over the phases are calculated in appendix D, see equations (D.6)–(D.12). These rather cumbersome expressions are substantially simplified after the integration over the Grassmann variables (see appendix E):

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q\iiint_0^{2\pi}\frac{\mathrm{d}\phi_{p,q,m}}{(2\pi)^3}(\mathcal{S}_{pq})^{k_1}(\mathcal{S}_{pm})^{k_2}(\mathcal{S}_{qm})^{k_3}$$
$$=\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R\eta_A^*\eta_A)_p\iiint_0^{2\pi}\frac{\mathrm{d}\phi_{p,q,m}}{(2\pi)^3}(\mathcal{S}_{pq})^{k_1}(\mathcal{S}_{pm})^{k_2}(\mathcal{S}_{qm})^{k_3}$$
$$=\frac{1}{32}\Xi(k_1,k_2,k_3)\frac{S_p^{k_1+k_2}}{\Gamma(k_1+k_2-1)}\frac{S_q^{k_1+k_3}}{\Gamma(k_1+k_3-1)}\frac{S_m^{k_2+k_3}}{\Gamma(k_2+k_3-1)}; \tag{67}$$

where

$$\Xi(k_1,k_2,k_3)=\frac{\Gamma(k_1-1/2)}{\pi^{1/2}k_1!}\frac{\Gamma(k_2-1/2)}{\pi^{1/2}k_2!}\frac{\Gamma(k_3-1/2)}{\pi^{1/2}k_3!}$$
$$\times(2k_1k_2k_3-k_1k_2-k_1k_3-k_2k_3)\Gamma(k_1+k_2+k_3-1). \tag{68}$$

After the integration over S-variables, the power series for $\mathcal{G}^{(3)}$ take the following form:

$$\left\langle\left\langle \mathcal{G}^{(3)}_{p\neq q}\right\rangle\right\rangle \simeq -\frac{\imath\pi^2}{4\sqrt{3}}\frac{\Omega}{\Delta}\sum_{\{m\neq p,q\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}\left(\frac{8b_{pq}}{\Omega^2}\right)^{k_1}\left(\frac{8b_{pm}}{\Omega^2}\right)^{k_2}\left(\frac{8b_{qm}}{\Omega^2}\right)^{k_3}$$
$$\times\Xi(k_1,k_2,k_3)(k_1+k_2-1)(k_1+k_3-1), \tag{69}$$

$$\left\langle\left\langle \mathcal{G}^{(3)}_{pp}\right\rangle\right\rangle \simeq -\frac{\imath\pi^2}{8\sqrt{3}}\frac{\Omega}{\Delta}\sum_{\{m,n\neq p\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}\left(\frac{8b_{pm}}{\Omega^2}\right)^{k_1}\left(\frac{8b_{pn}}{\Omega^2}\right)^{k_2}\left(\frac{8b_{mn}}{\Omega^2}\right)^{k_3}$$
$$\times\Xi(k_1,k_2,k_3)(k_1+k_2)(k_1+k_2-1). \tag{70}$$

Following the procedure described in the preceding section, we Fourier transform the real part of equations (69) and (70) and obtain

$$\left\langle\left\langle \mathcal{G}^{(3)}_{p\neq q}(t)\right\rangle\right\rangle \simeq \frac{\pi^3}{4\sqrt{3}}\frac{1}{(t\Delta)^2}\sum_{\{m\neq p,q\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}(-8b_{pq}t^2)^{k_1}(-8b_{pm}t^2)^{k_2}(-8b_{qm}t^2)^{k_3}$$
$$\times\frac{\Xi(k_1,k_2,k_3)}{\Gamma(2[k_1+k_2+k_3]-1)}(k_1+k_2-1)(k_1+k_3-1), \tag{71}$$

$$\left\langle\left\langle \mathcal{G}^{(3)}_{pp}(t)\right\rangle\right\rangle \simeq \frac{\pi^3}{8\sqrt{3}}\frac{1}{(t\Delta)^2}\sum_{\{m,n\neq p\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}(-8b_{pm}t^2)^{k_1}(-8b_{pn}t^2)^{k_2}(-8b_{mn}t^2)^{k_3} \times \frac{\Xi(k_1,k_2,k_3)}{\Gamma(2[k_1+k_2+k_3]-1)}(k_1+k_2)(k_1+k_2-1). \tag{72}$$

The triple sums on the rhs of equations (71) and (72) cannot be reduced to a product of simple sums. Therefore, the summation over $k_{1,2,3}$ is not trivial [6]. To verify SuSyFT, we calculate the contribution of three interacting matrices to the form factor (58). The answer looks more compact if we at first symmetrize the expression for $\mathcal{G}^{(3)}(t)$ with respect to $k_{1,2,3}$ and then turn to the ordered sum over three remaining indices:

$$K^{(3)}(t) \simeq \frac{2}{\sqrt{3}t^2}\frac{1}{N}\sum_{\{m>n>p\}}^{N}\sum_{k_{1,2,3}=0}^{\infty}(-2b_{pm}t^2)^{k_1}(-2b_{pn}t^2)^{k_2}(-2b_{mn}t^2)^{k_3} \times \frac{4^{k_1+k_2+k_3-1}\pi\,\Xi(k_1,k_2,k_3)}{\Gamma(2[k_1+k_2+k_3]-1)}(k_1+k_2+k_3-1)(k_1+k_2+k_3-3/2). \tag{73}$$

Equation (73) coincides with the expression for $K^{(3)}(t)$ obtained by TVE [4].

We recall that all results of this section should be taken into account in the VE if $\omega \ll \mathcal{B} \ll 1$ and the assumptions of the large-scale limit hold true, i.e., the sums over m and n converge at the large spatial scales. This is the case, for instance, for the spectral statistics of the critical PLBRMs where the relevant energy range is small $\omega < \mathcal{B}\Delta$ while the characteristic spatial scale, which governs the two-level correlations in the framework of 2- and 3-matrix approximation, is large $X_c \sim \mathcal{B}/\omega \gg 1$ [6].

5.3. Verification of the supersymmetric VE

Let us verify that the power series obtained for $\mathcal{G}^{(m)}$, $m \geqslant 2$, really obey the estimate (27). If we consider a term of the power series with given powers k_j, integrate it over all R-variables in the saddle-point approximation, scale S-variables by Ω and perform the summation over all internal indices $m, n, \ldots \neq p, q$, then a simple power counting shows that the answer will be proportional to

$$\frac{\Omega^m}{\Omega^2}\left(\frac{\mathcal{B}}{\Omega}\right)^{2(k_1+k_2+\cdots)}$$

(cf equations (52), (53) and (69), (70)). Here Ω^m in the numerator and Ω^2 in the denominator result from the integration measure and from the factor $\mathcal{RA}$, respectively. We can rewrite this ratio as follows:

$$\mathcal{B}^{m-2}\left(\frac{\mathcal{B}}{\Omega}\right)^{2(k_1+k_2+\cdots)-(m-2)}.$$

Obviously, having performed the summation over k_j, the answer for $\mathcal{G}^{(m)}$ can be written as a product

$$\mathcal{G}^{(m)}(\mathcal{B},\Omega) = \mathcal{B}^{m-2}\bar{\mathcal{G}}^{(m)}\left(\frac{\Omega}{\mathcal{B}}\right), \tag{74}$$

which agrees with the estimate (27) and with formula (40).

We recall that the mth virial coefficient $\bar{\mathcal{G}}^{(m)}$ depends on the parameter $\Omega/\mathcal{B}$, which can take on an arbitrary value and is not assumed to be either small or large. The successive terms

of the VE decrease with increasing m only if the absolute value of the virial coefficients is bounded for the arbitrary ratio $\Omega/\mathcal{B}$.

6. Conclusions

In the present work, we develop a supersymmetric field-theoretical description of a Gaussian ensemble of the almost diagonal Hermitian random matrices. In this ensemble, the off-diagonal matrix elements are assumed to be parametrically smaller than the diagonal ones: $H_{ii} \sim 1, H_{ij}/H_{ii} \sim \mathcal{B} \ll 1$. We use the method of the supersymmetry to perform an ensemble averaging. The standard route of the derivation of the supersymmetric nonlinear σ-model cannot be taken in this case, since the diffusion approximation fails.

As an alternative to the supersymmetric σ-model, we derive a virial expansion (VE) in the number of 'interacting' supermatrices, which is controlled by the small parameter $\mathcal{B}$. Each supermatrix can be related to a localized eigenstate of the diagonal part of RMs. Thus, the supermatrix interaction describes the interaction of the localized wavefunctions via the (small) off-diagonal elements of RMs. The principal idea of VE is similar to one used in VE based on the Trotter formula [4]. Nevertheless, the supersymmetric VE is much more powerful since it allows us to study not only the spectral correlations but also the correlation of wavefunctions taken at different energies and in different space points.

The application of supersymmetric VE becomes especially efficient in a situation, when (i) the relevant energy range is much smaller than the typical value of the diagonal elements of RMs, $\omega \ll 1$ and (ii) the large-scale approximation can be used, see section 4. In this case, the massive degrees of freedom are integrated out by the saddle-point approximation. This step is a counterpart of the saddle-point approximation used in the derivation of the nonlinear σ-model. However, the saddle-point approximation in the VE requires only the linear constraint: $\mathrm{Str}[Q] = 0$.

One of the main results of the paper is the integral expression for the mth term of VE, equations (38) and (39), which is governed by the interaction of m supermatrices. The superintegrals in this formula completely circumvent a complicated combinatorial calculations in the theory based on the Trotter formula. We note in passing that in this way we manage to reduce the complicated problem of simultaneous colouring of edges of several graphs (which is along standing problem in the statistical physics and the applied mathematics) to the calculation of superintegrals. The superintegrals are calculated explicitly for the cases of 2- and 3-matrix interaction with the help of the parameterization suggested in [20]. The results containing in equations (52), (53), (60), (61) and (69), (70) have been obtained for the first time. They have been derived for a generic ensemble of the almost diagonal RMs described by equation (1) in the case of the unitary symmetry class. We note that our approach can be easily generalized to the other symmetry classes.

The virial expansion generates a regular perturbation theory in powers of $\mathcal{B}$ for a variety of the correlation functions in the different models of the almost diagonal RMs. Applications to certain RMT models will be presented elsewhere. The critical ensemble of RMs [6] and the Moshe–Neuberger–Shapiro model [15] with the orthogonal symmetry are two examples of promising applications. The next important step would be derivation of non-perturbative results directly from the supersymmetric action in the large-scale approximation. The non-perturbative solutions could particularly shed light on the following problem: under what circumstances an interaction between the localized states can lead to the criticality or to the delocalization. This question is of fundamental importance in the theory of disordered [24] and strongly correlated disordered systems [25].

Acknowledgments

The authors are very grateful to Vladimir Kravtsov for initiating the project on the field theory representation of almost diagonal RMs and critical comments concerning the accuracy of VE and to Vladimir Yudson for useful discussions. AO acknowledges support by the Dutch Science Foundation NWO/FOM.

Appendix A. Parametrization of matrix Q in the retarded-advanced sector

Let us find a parameterization of the matrix Q defined as direct product of the supervector by the conjugated supervector

$$Q \equiv \Psi \otimes \bar{\Psi} = \begin{pmatrix} \Psi_R \otimes \Psi_R^\dagger & \Psi_R \otimes K\Psi_A^\dagger \\ \Psi_A \otimes \Psi_R^\dagger & \Psi_A \otimes K\Psi_A^\dagger \end{pmatrix}, \tag{A.1}$$

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_A \end{pmatrix}, \qquad \Psi_{R/A} = \begin{pmatrix} s_{R/A} \\ \chi_{R/A} \end{pmatrix}, \qquad K \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.2}$$

where indices R and A are referring to the retarded and advanced sectors correspondingly.

Matrix $Q_{RR} = \Psi_R \otimes \Psi_R^\dagger$ is the orthogonal projector on vector Ψ_R and can be diagonalized by the unitary matrix U_R:

$$\begin{aligned} &Q_{RR} = U_R D_{RR} U_R^{-1} \\ &U_R = \begin{pmatrix} 1 - \frac{1}{2}\eta_R^*\eta_R & -\eta_R^* \\ \eta_R & 1 + \frac{1}{2}\eta_R^*\eta_R \end{pmatrix}, \\ &U_R^{-1} = \begin{pmatrix} 1 - \frac{1}{2}\eta_R^*\eta_R & \eta_R^* \\ -\eta_R & 1 + \frac{1}{2}\eta_R^*\eta_R \end{pmatrix}, \qquad D_{RR} = \begin{pmatrix} \lambda_R^2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{A.3}$$

where $\eta_R = \chi_R/s_R, \lambda_R^2 = \|\Psi_R\|^2 = |s_R|^2 + \chi_R^*\chi_R$. In a similar way block Q_{AA} can be diagonalized by the pseudounitary matrix U_A $\left(U_A^\dagger K U_A = K\right)$:

$$\begin{aligned} &Q_{AA} = U_A D_{AA} U_A^{-1} \\ &U_A = \begin{pmatrix} 1 + \frac{1}{2}\eta_A^*\eta_A & \eta_A^* \\ \eta_A & 1 - \frac{1}{2}\eta_A^*\eta_A \end{pmatrix}, \qquad U_A^{-1} = \begin{pmatrix} 1 + \frac{1}{2}\eta_A^*\eta_A & -\eta_A^* \\ -\eta_A & 1 - \frac{1}{2}\eta_A^*\eta_A \end{pmatrix}, \\ &D_{AA} = \begin{pmatrix} -\lambda_A^2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{A.4}$$

with $\eta_A = \chi_A/s_A, \lambda_A^2 = -\|\Psi_A\|^2 = |s_A|^2 - \chi_A^*\chi_A$. Moreover, U_R, U_A diagonalize off-diagonal blocks of Q:

$$\begin{aligned} &Q_{AR} = U_A D_{AR} U_R^{-1} \qquad Q_{RA} = U_R D_{RA} U_A^{-1} \\ &D_{AR} = \begin{pmatrix} e^{i\phi}\lambda_R\lambda_A & 0 \\ 0 & 0 \end{pmatrix} \qquad D_{RA} = \begin{pmatrix} -e^{-i\phi}\lambda_R\lambda_A & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{A.5}$$

where $e^{i\phi} = s_R^* s_A/(|s_R s_A|)$. Thus, the matrix Q can be parameterized as follows:

$$\begin{aligned} &Q = UDU^{-1} \\ &U = \begin{pmatrix} U_R & 0 \\ 0 & U_A \end{pmatrix}, \qquad D = \begin{pmatrix} D_{RR} & D_{RA} \\ D_{AR} & D_{AA} \end{pmatrix} = \begin{pmatrix} \lambda_R^2 & 0 & -e^{-i\phi}\lambda_R\lambda_A & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\phi}\lambda_R\lambda_A & 0 & -\lambda_A^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A.6}$$

The measure in this parameterization can be easily found by calculating the Jacobian (Berezenian) of the transformation (A.6) and is equal to

$$\mathcal{D}\{Q\} = \frac{2}{\pi}\frac{\mathrm{d}\lambda_R\,\mathrm{d}\lambda_A}{\lambda_R\lambda_A}\,\mathrm{d}\phi\,\mathrm{d}\eta_R^*\,\mathrm{d}\eta_R\,\mathrm{d}\eta_A^*\,\mathrm{d}\eta_A. \tag{A.7}$$

Using equation (A.6) and taking into account the rotational symmetry of the supertrace, we obtain

$$\mathrm{Str}(Q) = R, \mathrm{Str}(Q^2) = R^2; \qquad R \equiv \lambda_R^2 - \lambda_A^2; \tag{A.8}$$

$$\mathrm{Str}(\Lambda Q) = S; \qquad S \equiv \lambda_R^2 + \lambda_A^2; \tag{A.9}$$

$$\begin{aligned}\mathrm{Str}[Q\tilde{Q}] = {} & \lambda_R^2\tilde{\lambda}_R^2(1-\alpha_R^*\alpha_R) + \lambda_A^2\tilde{\lambda}_A^2(1+\alpha_A^*\alpha_A) \\ & - 2\cos\theta\lambda_R\tilde{\lambda}_R\lambda_A\tilde{\lambda}_A\left(1-\tfrac{1}{2}\alpha_R^*\alpha_R\right)\left(1+\tfrac{1}{2}\alpha_A^*\alpha_A\right);\end{aligned} \tag{A.10}$$

where $\alpha_{R/A} \equiv \eta_{R/A} - \tilde{\eta}_{R/A}$; $\theta \equiv \phi - \tilde{\phi} + \Delta$; $\Delta = \frac{\mathrm{i}}{2}(\tilde{\eta}_R^*\eta_R - \eta_R^*\tilde{\eta}_R + \tilde{\eta}_A^*\eta_A - \eta_A^*\tilde{\eta}_A)$ and tilde marks the variables of the matrix $\tilde{Q}$. The last identity (A.10) follows from the well-known property [27] of the matrices U:

$$U_R^{-1}(\tilde{\eta}_R)U_R(\eta_R) = U_R(\eta_R - \tilde{\eta}_R)\exp\left(\tfrac{1}{2}(\tilde{\eta}_R^*\eta_R - \eta_R^*\tilde{\eta}_R)\right), \tag{A.11}$$

$$U_A^{-1}(\tilde{\eta}_A)U_A(\eta_A) = U_A(\eta_A - \tilde{\eta}_A)\exp\left(-\tfrac{1}{2}(\tilde{\eta}_A^*\eta_A - \eta_A^*\tilde{\eta}_A)\right). \tag{A.12}$$

Appendix B. Parametrization of matrix Q in the retarded-retarded sector

In the retarded-retarded sector the matrix Q is defined similar to equation (A.1) but without matrix K:

$$Q \equiv \Psi \otimes \Psi^\dagger = \begin{pmatrix} \Psi_R \otimes \Psi_{R'}^\dagger & \Psi_R \otimes \Psi_{R'}^\dagger \\ \Psi_{R'} \otimes \Psi_R^\dagger & \Psi_{R'} \otimes \Psi_{R'}^\dagger \end{pmatrix}, \tag{B.1}$$

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_{R'} \end{pmatrix}, \qquad \Psi_{R/R'} = \begin{pmatrix} s_{R/R'} \\ \chi_{R/R'} \end{pmatrix}, \tag{B.2}$$

where indices R and R' are referring to the retarded sectors of two different Green's functions. As a result, one can diagonalize Q by transformation similar to equation (A.6):

$$\begin{aligned} & Q = UDU^{-1} \\ & U = \begin{pmatrix} U_R & 0 \\ 0 & U_{R'} \end{pmatrix}, \qquad D = \begin{pmatrix} D_{RR} & D_{RR'} \\ D_{R'R} & D_{R'R'} \end{pmatrix} = \begin{pmatrix} \lambda_R^2 & 0 & \mathrm{e}^{-\mathrm{i}\phi}\lambda_R\lambda_{R'} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{e}^{\mathrm{i}\phi}\lambda_R\lambda_{R'} & 0 & \lambda_{R'}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{B.3}$$

where U_R is defined in equation (A.3) and $U_{R'}$ is obtained from U_R by replacing subscript R by R' everywhere. All parameters appearing in (B.3) are defined in the same way as before:

$$\begin{aligned} & \eta_R = \frac{\chi_R}{s_R}, \qquad \lambda_R^2 = |s_R|^2 + \chi_R^*\chi_R, \qquad \mathrm{e}^{\mathrm{i}\phi} = \frac{s_R^* s_{R'}}{|s_R s_{R'}|}, \\ & \eta_{R'} = \frac{\chi_{R'}}{s_{R'}}, \qquad \lambda_{R'}^2 = |s_{R'}|^2 + \chi_{R'}^*\chi_{R'}. \end{aligned} \tag{B.4}$$

The measure in this parameterization remains the same as in equation (A.7), while the roles of variables R and S defined in equations (A.8) and (A.9) are interchanged now:

$$\mathrm{Str}(Q) = S, \qquad \mathrm{Str}(Q^2) = S^2; \qquad S \equiv \lambda_R^2 + \lambda_{R'}^2; \tag{B.5}$$

$$\mathrm{Str}(\Lambda Q) = R; \qquad R \equiv \lambda_R^2 - \lambda_{R'}^2. \tag{B.6}$$

Finally, the expression for $\mathrm{Str}[Q\tilde{Q}]$ is again similar to equation (A.10):

$$\begin{aligned}\mathrm{Str}[Q\tilde{Q}] = {} & \lambda_R^2\tilde{\lambda}_R^2(1-\alpha_R^*\alpha_R) + \lambda_{R'}^2\tilde{\lambda}_{R'}^2(1-\alpha_{R'}^*\alpha_{R'}) \\ & - 2\cos\theta\lambda_R\tilde{\lambda}_R\lambda_{R'}\tilde{\lambda}_{R'}\left(1-\tfrac{1}{2}\alpha_R^*\alpha_R\right)\left(1-\tfrac{1}{2}\alpha_{R'}^*\alpha_{R'}\right);\end{aligned} \tag{B.7}$$

where $\alpha_{R/R'} \equiv \eta_{R/R'} - \tilde{\eta}_{R/R'}$; $\theta \equiv \phi - \tilde{\phi} + \Delta$; $\Delta = \frac{\mathrm{i}}{2}(\tilde{\eta}_R^*\eta_R - \eta_R^*\tilde{\eta}_R - \tilde{\eta}_{R'}^*\eta_{R'} + \eta_{R'}^*\tilde{\eta}_{R'})$.

Appendix C. Integrals for integer powers of sin($\phi/2$)

One needs the following formulae to average a product of the supertraces over the phases:

$$\mathcal{F}(k) \equiv \frac{1}{2\pi}\int_0^{2\pi} \sin^{2k}\left(\frac{\phi}{2}\right)\mathrm{d}\phi = \frac{\Gamma(k+1/2)}{\pi^{1/2}\Gamma(k+1)}; \tag{C.1}$$

$$\begin{aligned}\mathcal{F}(k_1,k_2,k_3) \equiv {} & \frac{1}{(2\pi)^2}\int\!\!\int_0^{2\pi} \sin^{2k_1}\left(\frac{\phi_1}{2}\right)\sin^{2k_2}\left(\frac{\phi_2}{2}\right)\sin^{2k_3}\left(\frac{\phi_1-\phi_2}{2}\right)\mathrm{d}\phi_{1,2} \\ = {} & \frac{\Gamma(k_1+1/2)}{\pi^{1/2}}\frac{\Gamma(k_2+1/2)}{\pi^{1/2}}\frac{\Gamma(k_3+1/2)}{\pi^{1/2}} \\ & \times \frac{\Gamma(k_1+k_2+k_3+1)}{\Gamma(k_1+k_2+1)\Gamma(k_1+k_3+1)\Gamma(k_2+k_3+1)}.\end{aligned} \tag{C.2}$$

We consider only integer powers of sines k and $k_{1,2,3}$. Equation (C.1) can be found in standard mathematical tables [28] while equation (C.2) can be proven by the induction over one of the exponents, for example, over k_3. Clearly, $\mathcal{F}(k,0,0) = \mathcal{F}(k)$; $\mathcal{F}(k_1,k_2,0) = \mathcal{F}(k_1)\mathcal{F}(k_2)$. This constitutes the induction basis. To check the hypothesis for arbitrary k_3, we assume that $\mathcal{F}(k_1',k_2',k_3)$ is known for arbitrary $k_{1,2}'$ and find a relation between $\mathcal{F}(k_1,k_2,k_3+1)$ and $\mathcal{F}(k_1',k_2',k_3)$. Simple trigonometric transformations together with integrations by parts yield

$$\begin{aligned}\mathcal{F}(k_1,k_2,k_3+1) = {} & \mathcal{F}(k_1+1,k_2,k_3) + \mathcal{F}(k_1,k_2+1,k_3) - 2\mathcal{F}(k_1+1,k_2+1,k_3) \\ & + \frac{2}{(k_1+1)(k_2+1)}\mathcal{F}^{(2)}(k_1+1,k_2+1,k_3),\end{aligned} \tag{C.3}$$

where

$$\mathcal{F}^{(p)}(k_1,k_2,k_3) \equiv \frac{1}{(2\pi)^2}\int\!\!\int_0^{2\pi}\left\{\partial_{\phi_1}^p\left[\sin^{2k_1}\left(\frac{\phi_1}{2}\right)\right]\right\}\sin^{2k_2}\left(\frac{\phi_2}{2}\right)\sin^{2k_3}\left(\frac{\phi_1-\phi_2}{2}\right), \tag{C.4}$$

$$\mathcal{F}^{(2)}(k_1,k_2,k_3) = \frac{k_1}{2}[(2k_1-1)\mathcal{F}(k_1-1,k_2,k_3) - 2k_1\mathcal{F}(k_1,k_2,k_3)]. \tag{C.5}$$

Inserting (C.5) into (C.3), we get the relation

$$\begin{aligned}\mathcal{F}(k_1,k_2,k_3+1) = {} & \mathcal{F}(k_1+1,k_2,k_3) + \left(1+\frac{2k_1+1}{k_2+1}\right)\mathcal{F}(k_1,k_2+1,k_3) \\ & - 2\left(1+\frac{k_1+1}{k_2+1}\right)\mathcal{F}(k_1+1,k_2+1,k_3).\end{aligned} \tag{C.6}$$

We substitute equation (C.2) into the right-hand side of equation (C.6) and derive the answer

$$\mathcal{F}(k_1,k_2,k_3+1)=\frac{\Gamma(k_1+1/2)}{\pi^{1/2}}\frac{\Gamma(k_2+1/2)}{\pi^{1/2}}\frac{\Gamma(k_3+3/2)}{\pi^{1/2}}\times\frac{\Gamma(k_1+k_2+k_3+2)}{\Gamma(k_1+k_2+1)\Gamma(k_1+k_3+2)\Gamma(k_2+k_3+2)},\tag{C.7}$$

which satisfies equation (C.2). Thus, the induction over k_3 is verified and equation (C.2) is proven.

Using (C.2), (C.4) and (C.5), one can also show that

$$\mathcal{F}^{(2n-1)}(k_1,k_2,k_3)=0,\qquad n=1,2,\ldots\tag{C.8}$$

$$\mathcal{F}^{(2)}(k_1,k_2,k_3)=\mathcal{F}(k_1,k_2,k_3)\frac{k_1k_2k_3}{k_1+k_2+k_3},\tag{C.9}$$

and

$$\begin{aligned}\mathcal{F}^{(4)}(k_1,k_2,k_3)&=\frac{k_1}{8}\big[2(k_1-1)(2k_1-1)(2k_1-3)\mathcal{F}(k_1-2,k_2,k_3)\\&\quad-4\big(4k_1^3-6k_1^2+4k_1-1\big)\mathcal{F}(k_1-1,k_2,k_3)+8k_1^3\mathcal{F}(k_1,k_2,k_3)\big]\\&=-\mathcal{F}^{(2)}(k_1,k_2,k_3)\left(1-\frac{(k_1-1)(k_2-1)(k_3-1)}{k_1+k_2+k_3-1}\right).\end{aligned}\tag{C.10}$$

Appendix D. Averaging $\mathrm{Str}^k[Q_1Q_2]$ and $\mathrm{Str}^{k_1}[Q_1Q_2]\mathrm{Str}^{k_2}[Q_1Q_3]\mathrm{Str}^{k_3}[Q_2Q_3]$ over the phases at $R_{1,2,3}=0$ and integers powers $k, k_{1,2,3}$

Let us average $\mathcal{S}_{12}^k\equiv(\mathrm{Str}[Q_1Q_2])^k|_{R_{1,2}=0}$ over the phases $\phi_{1,2}$:

$$\mathcal{S}_{12}^k=\left[\frac{S_1S_2}{4}\left(4\sin^2\left(\frac{\phi}{2}\right)+2(\alpha_A^*\alpha_A-\alpha_R^*\alpha_R)\sin^2\left(\frac{\phi}{2}\right)+\frac{\cos(\phi)}{2}\alpha_A^*\alpha_A\alpha_R^*\alpha_R\right)\right]^k;\tag{D.1}$$

where k is integer; $\phi\equiv\phi_{12}+\Delta_{12}$; $\phi_{12}\equiv\phi_1-\phi_2$ and indices 1 and 2 mark the variables of the matrices Q_1 and Q_2, respectively (see also notations in appendix A). We have to calculate the following integral of the periodic function:

$$\frac{1}{(2\pi)^2}\int\!\!\int_0^{2\pi}\mathrm{d}\phi_{1,2}\mathcal{S}_{12}^k(\phi_1-\phi_2+\Delta_{12})=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi_{12}\mathcal{S}_{12}^k(\phi_{12}).\tag{D.2}$$

The nilpotents from $\Delta_{12}\equiv\frac{\imath}{2}\big(\Delta_R^{(12)}+\Delta_A^{(12)}\big)$, $\Delta_{R/A}^{(12)}=(\eta_{R/A})_2^*(\eta_{R/A})_1$ − c.c. give no contribution to the integral (D.2) due to the periodicity of the integrand.

Collecting the terms with the same powers of the Grassmann variables we find

$$\begin{aligned}\mathcal{S}_{12}^k&=\left(\frac{S_1S_2}{4}\right)^k\left\{(2\sin(\phi/2))^{2k}+\frac{k}{2}(2\sin(\phi/2))^{2k}(\alpha_A^*\alpha_A-\alpha_R^*\alpha_R)\right.\\&\quad\left.+\frac{k}{2}\left[(2\sin(\phi/2))^{2(k-1)}-\frac{k}{2}(2\sin(\phi/2))^{2k}\right]\alpha_A^*\alpha_A\alpha_R^*\alpha_R\right\}.\end{aligned}\tag{D.3}$$

We insert equation (D.3) into the integral (D.2) and integrate over the phase ϕ_{12} using equation (C.1):

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi_{12}\mathcal{S}_{12}^k(\phi_{12})&=\left(\frac{S_1S_2}{4}\right)^k\frac{\Gamma(2k-1)}{\Gamma(k-1)\Gamma(k)}\\&\quad\times\left\{\frac{2k-1}{k(k-1)}(2+k(\alpha_A^*\alpha_A-\alpha_R^*\alpha_R))-k\alpha_A^*\alpha_A\alpha_R^*\alpha_R\right\}.\end{aligned}\tag{D.4}$$

Note that the equality (D.3) can be rewritten in a more compact form:

$$\mathcal{S}_{12}^{k} = (S_1 S_2)^k \left\{ (\sin(\phi/2))^{2k}\, \mathrm{e}^{k\Upsilon_{12}} - \frac{k}{4}(\sin(\phi/2))^{2(k-1)}\Upsilon_{12}^2 \right\}; \qquad \Upsilon_{12} \equiv \frac{\alpha_A^* \alpha_A - \alpha_R^* \alpha_R}{2}. \tag{D.5}$$

Unlike equation (D.2) written for the case of two linked supermatrices, the averaged product $\mathcal{S}_{12}^{k_1}\mathcal{S}_{13}^{k_2}\mathcal{S}_{23}^{k_3}$ (which includes three linked supermatrices) depends on the nilpotents coming from $\Delta^{(3)} \equiv \Delta_{12} + \Delta_{13} + \Delta_{23}$ and it can be expanded in the even powers of $\Delta^{(3)}$:

$$\frac{1}{(2\pi)^3}\iiint_0^{2\pi} \mathrm{d}\phi_{1,2,3}\mathcal{S}_{12}^{k_1}(\phi_1 - \phi_2 + \Delta_{12})\mathcal{S}_{13}^{k_2}(\phi_1 - \phi_3 + \Delta_{13})\mathcal{S}_{23}^{k_3}(\phi_2 - \phi_3 + \Delta_{23})$$
$$= \frac{1}{(2\pi)^2}\iint_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi' \mathcal{S}_{12}^{k_1}(\phi)\mathcal{S}_{13}^{k_2}(\phi')\mathcal{S}_{23}^{k_3}(\phi - \phi' + \Delta^{(3)}) \tag{D.6}$$
$$= \sum_{p=0}^{3} \frac{(\Delta^{(3)})^{2p}}{(2p)!}\left[\frac{1}{(2\pi)^2}\iint_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi' \mathcal{S}_{12}^{k_1}(\phi)\mathcal{S}_{13}^{k_2}(\phi')\partial_\phi^{2p}\mathcal{S}_{23}^{k_3}(\phi - \phi')\right]. \tag{D.7}$$

Here $k_{1,2,3}$ are integer. There are no odd powers of $\Delta^{(3)}$ due to the equality (C.8). For the purpose of the present paper, we need only the terms with $p = 0, 1, 2$. The term $\propto (\Delta^{(3)})^6$ yields zero after multiplication by the factor $\mathcal{R}_p\mathcal{A}_q$ (see equations (39) and (38)). Using formula (D.5) and the results of appendix C, we find

$$\frac{(\Delta^{(3)})^{2p}}{(2p)!}\left[\frac{1}{(2\pi)^2}\iint_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi'\, \mathcal{S}_{12}^{k_1}(\phi)\mathcal{S}_{13}^{k_2}(\phi')\partial_\phi^{2p}\mathcal{S}_{23}^{k_3}(\phi - \phi')\right]\Bigg|_{p=2}$$
$$= (S_1 S_2)^{k_1}(S_1 S_3)^{k_2}(S_2 S_3)^{k_3}\frac{(\Delta^{(3)})^4}{4!}\mathcal{F}^{(4)}(k_1, k_2, k_3); \tag{D.8}$$
$$\frac{(\Delta^{(3)})^{2p}}{(2p)!}\left[\frac{1}{(2\pi)^2}\iint_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi'\, \mathcal{S}_{12}^{k_1}(\phi)\mathcal{S}_{13}^{k_2}(\phi')\partial_\phi^{2p}\mathcal{S}_{23}^{k_3}(\phi - \phi')\right]\Bigg|_{p=1}$$
$$= (S_1 S_2)^{k_1}(S_1 S_3)^{k_2}(S_2 S_3)^{k_3}\frac{(\Delta^{(3)})^2}{8}\big(2\mathcal{F}^{(2)}(k_1, k_2, k_3)(k_1\Upsilon_{12} + k_2\Upsilon_{13} + k_3\Upsilon_{23})^2$$
$$- \big[\mathcal{F}^{(2)}(k_1 - 1, k_2, k_3)k_1\Upsilon_{12}^2 + \mathcal{F}^{(2)}(k_1, k_2 - 1, k_3)k_2\Upsilon_{13}^2$$
$$+ \mathcal{F}^{(2)}(k_1, k_2, k_3 - 1)k_3\Upsilon_{23}^2\big]\big); \tag{D.9}$$
$$\frac{(\Delta^{(3)})^{2p}}{(2p)!}\left[\frac{1}{(2\pi)^2}\iint_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi'\, \mathcal{S}_{12}^{k_1}(\phi)\mathcal{S}_{13}^{k_2}(\phi')\partial_\phi^{2p}\mathcal{S}_{23}^{k_3}(\phi - \phi')\right]\Bigg|_{p=0}$$
$$= (S_1 S_2)^{k_1}(S_1 S_3)^{k_2}(S_2 S_3)^{k_3}(\mathrm{Part}_1 + \mathrm{Part}_2 + \mathrm{Part}_3); \tag{D.10}$$

where

$$\mathrm{Part}_1 = \frac{\mathcal{F}(k_1, k_2, k_3)}{4}(2k_1k_2k_3\Upsilon_{1,2}\Upsilon_{1,3}\Upsilon_{2,3}[k_1\Upsilon_{1,2} + k_2\Upsilon_{1,3} + k_3\Upsilon_{2,3}]$$
$$+ (k_1\Upsilon_{1,2}k_2\Upsilon_{1,3})^2 + (k_1\Upsilon_{1,2}k_3\Upsilon_{2,3})^2 + (k_2\Upsilon_{1,3}k_3\Upsilon_{2,3})^2),$$
$$\mathrm{Part}_2 = -\frac{1}{8}\big(\mathcal{F}(k_1 - 1, k_2, k_3)k_1\Upsilon_{1,2}^2(k_2\Upsilon_{1,3} + k_3\Upsilon_{2,3})^2 \tag{D.11}$$
$$+ \mathcal{F}(k_1, k_2 - 1, k_3)k_2\Upsilon_{1,3}^2(k_1\Upsilon_{1,2} + k_3\Upsilon_{2,3})^2$$
$$+ \mathcal{F}(k_1, k_2, k_3 - 1)k_3\Upsilon_{2,3}^2(k_1\Upsilon_{1,2} + k_2\Upsilon_{1,3})^2\big),$$

$$\mathrm{Part}_3 = \frac{1}{16}\big(\mathcal{F}(k_1-1,k_2-1,k_3)k_1\Upsilon_{1,2}^2k_2\Upsilon_{1,3}^2 + \mathcal{F}(k_1-1,k_2,k_3-1)k_1\Upsilon_{1,2}^2k_3\Upsilon_{2,3}^2 + \mathcal{F}(k_1,k_2-1,k_3-1)k_2\Upsilon_{1,3}^2k_3\Upsilon_{2,3}^2\big). \tag{D.12}$$

Appendix E. Results of the integration over the Grassmann variables

A direct integration over the large number of the Grassmann variables (12 Grassmanns in the case of 3-matrix approximation) is technically trivial but very long and boring arithmetic procedure. We have used the 'Grassmann' package of the 'Maple' system to do this step of the calculations. Here, we present the results of this procedure which are necessary for the calculations in the 3-matrix approximation

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q(\Delta^{(3)})^4 = \frac{4!}{2^4}; \tag{E.1}$$

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q(\Delta^{(3)})^2 \times \begin{Bmatrix}\Upsilon_{p,q}^2\\ \Upsilon_{p,m}^2\\ \Upsilon_{q,m}^2\end{Bmatrix} = 0; \tag{E.2}$$

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q(\Delta^{(3)})^2 \times \begin{Bmatrix}\Upsilon_{p,q}\Upsilon_{q,m}\\ \Upsilon_{p,m}\Upsilon_{m,q}\\ \Upsilon_{q,p}\Upsilon_{p,m}\end{Bmatrix} = \frac{1}{2^2}; \tag{E.3}$$

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q \times \begin{Bmatrix}\Upsilon_{p,q}^2\Upsilon_{q,m}^2\\ \Upsilon_{p,m}^2\Upsilon_{m,q}^2\\ \Upsilon_{q,p}^2\Upsilon_{p,m}^2\end{Bmatrix} = \frac{1}{2^2}; \tag{E.4}$$

$$\int \mathrm{d}\{\eta_R^*\eta_R\eta_A^*\eta_A\}_{p,q,m}(\eta_R^*\eta_R)_p(\eta_A^*\eta_A)_q \times \begin{Bmatrix}\Upsilon_{p,q}^2\Upsilon_{q,m}\Upsilon_{m,p}\\ \Upsilon_{p,q}\Upsilon_{q,m}^2\Upsilon_{m,p}\\ \Upsilon_{p,q}\Upsilon_{q,m}\Upsilon_{m,p}^2\end{Bmatrix} = \frac{1}{2^2}; \tag{E.5}$$

see the definitions of the nilpotents in the previous appendix. Note that all three indices p, q and m are different.

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